Convergentes de Fracciones continuas

Seminario 1 de Teoria de Numeros

Alejandro Pallais Garcia

Universidad del Valle de Guatemala

6 de octubre de 2023

Contenido

Contenido

Fracciones continuas y sus convergentes

$\forall A \in \mathbb{Q} \exists a_0 \in \mathbb{Z}, \{a_i\}_{i=1}^{n\in\mathbb{Z}^+} \subseteq \mathbb{Z}^+ \ni$

$$A = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots a_{n-1} + \frac{1}{a_n}}}$$

para una notacion mas amigable se usa $A = [a_0; a_1, a_2, \cdots, a_n]$
Si $k \in \mathbb{N} \cap [0, n]$, el k-convergente de A es

$$C_k = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots a_{k-1} + \frac{1}{a_k}}} = [a_0; a_1, a_2, \cdots, a_k]$$

Teorema 15.2

Si $A = [a_0; a_1, \cdots, a_n] \in \mathbb{Q}$, $p_0 = a_0, p_1 = a_1 a_0 + 1, q_0 = 1, q_1 = a_1$ y

$$p_k = a_k p_{k-1} + p_{k-2} \text{ y } q_k = a_k q_{k-1} + q_{k-2}$$

$\forall k \in \mathbb{N} \cap [2, n] \Rightarrow$

$$C_k = \frac{p_k}{q_k}, \forall k \in \mathbb{N} \cap [0, n]$$

Contenido

resta de convergentes

$$C_{k+1} - C_k = \cfrac{1}{a_1 + \cfrac{1}{\ddots a_{k-1} + \cfrac{1}{a_k + \cfrac{1}{a_{k+1}}}}} - \cfrac{1}{a_1 + \cfrac{1}{\ddots a_{k-1} + \cfrac{1}{a_k}}}$$

$$C_{k+1} - C_k = \frac{P_{k+1}}{q_{k+1}} - \frac{p_k}{q_k} = \frac{p_{k+1} q_k - p_k q_{k+1}}{q_{k+1} q_k}$$

Teorema 15.3

$$p_{k+1}q_k - p_kq_{k+1} = (-1)^k, \forall k \in \mathbb{N} \cap [0, n-1]$$

Demostración.

Induccion sobre k

Si $k = 0 \Rightarrow$

$$p_1q_0 - p_0q_1 = (a_1a_0 + 1)1 - a_0a_1 = 1 = 1^0$$

si para $k = \alpha, p_{\alpha+1}q_\alpha - p_\alpha q_{\alpha+1} = (-1)^\alpha \Rightarrow$

$$p_{\alpha+2}q_{\alpha+1} - p_{\alpha+1}q_{\alpha+2} = (a_{\alpha+2}p_{\alpha+1} + p_\alpha)q_{\alpha+1} - p_{\alpha+1}(a_{\alpha+2}q_{\alpha+1} + q_\alpha) =$$

$$= a_{\alpha+2}(p_{\alpha+1}q_{\alpha+1} - p_{\alpha+1}q_{\alpha+1}) + p_\alpha q_{\alpha+1} - p_{\alpha+1}q_\alpha = -(-1)^\alpha = (-1)^{\alpha+1}$$

□

$\therefore C_{k+1} - C_k = \frac{(-1)^k}{q_{k+1}q_k}$

corolario

$$MCD(p_k, q_k) = 1, \forall k \in \mathbb{N} \cap [1, n]$$

Demostración.

$MCD(p_k, q_k) | p_k, q_k \Rightarrow MCD(p_k, q_k) | p_{k+1}q_k - p_k q_{k+1} = (-1)^k$
$\therefore 0 < MCD(p_k, q_k) = 1$ $\square$

$C_k = \frac{p_k}{q_k}$ son fracciones reducidas

ejemplo

Consideremos $[0; 1, 1, \cdots, 1]$

$$\frac{1}{1+\frac{1}{\ddots 1+\frac{1}{1+\frac{1}{1+\frac{1}{1+1}}}}} = \frac{1}{1+\frac{1}{\ddots 1+\frac{1}{1+\frac{1}{1+\frac{1}{2}}}}} = \frac{1}{1+\frac{1}{\ddots 1+\frac{1}{1+\frac{1}{\frac{2+1}{2}}}}} = \frac{1}{1+\frac{1}{\ddots 1+\frac{1}{1+\frac{2}{3}}}} =$$

$$= \frac{1}{1+\frac{1}{\ddots 1+\frac{1}{\frac{3+2}{3}}}} = \frac{1}{1+\frac{1}{\ddots 1+\frac{3}{5}}} = \frac{1}{1+\frac{1}{\ddots \frac{5+3}{5}}} = \frac{1}{1+\frac{1}{\ddots \frac{8}{5}}} = \cdots$$

ejemplo fibonacci

notemos que $p_2 = 0, p_1 = 1 * 0 + 1 = 1, p_2 = 1 * 1 + 0 = 1,$
$p_k = 1 * p_{k-1} + p_{k-2} = p_{k-1} + p_{k-2} \Rightarrow p_k = f_k$
analogamente $q_k = f_{k+1} \Rightarrow$

$$f_{k+1}^2 - f_k f_{k+2} = p_{k+1} q_k - p_k q_{k+1} = (-1)^k$$

ejemplo diofantinas

$$ax + by = c, MCD(a, b) = d|c$$

pero como $a/d, b/d, c/d \in \mathbb{Z}, MCD(a/d, b/d) = 1|(c/d) \Rightarrow$

$$ax + by = c, MCD(a, b) = 1$$

$\Rightarrow \exists x_o, y_o \in \mathbb{Z} \ni ax_0 + by_0 = 1 \Rightarrow a(x_0 c) + b(y_0 c) = c \Rightarrow$

$$ax + by = 1$$

ejemplo diofantinas

$$ax + by = 1, MCD(a, b) = 1$$

$\frac{a}{b} = [a_0 : a_1, \cdots a_n] \Rightarrow$

$$C_{n-1} = \frac{p_{n-1}}{q_{n-1}}, C_n = \frac{p_n}{q_n} = \frac{a}{b}$$

como $MCD(p_n, q_n) = 1 = MCD(a, b) \Rightarrow p_n = a, q_n = b \Rightarrow$

$$aq_{n-1} - bp_{n-1} = p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$$

Si n es impar

$$aq_{n-1} + b(-p_{n-1}) = 1$$

Si n es par

$$a(-q_{n-1}) + bp_{n-1} = 1$$

ejemplo $172x + 20y = 1000$

como $MCD(172, 20) = 4|1000 \Rightarrow 43x + 5y = 250 \Rightarrow$

$$43x + 5y = 1$$

$\Rightarrow MCD(43, 5)$

$$\begin{array}{rrr} 43 = & 8 \cdot 5 & +3 \\ 5 = & 1 \cdot 3 & +2 \\ 3 = & 1 \cdot 2 & +1 \\ 2 = & 2 \cdot 1 & +0 \end{array} \Rightarrow \begin{array}{rrr} 43/5 = & 8 & +3/5 \\ 5/3 = & 1 & +2/3 \\ 3/2 = & 1 & +1/2 \\ 2/1 = & 2 & \end{array}$$

$$\Rightarrow \frac{43}{5} = 8 + \frac{1}{5/3} = 8 + \frac{1}{1 + \frac{1}{3/2}} = 8 + \frac{1}{1 + \frac{1}{1 + \frac{1}{2}}} = [8; 1, 1, 2]$$

ejemplo

$43/5 = [8; 1, 1, 2]$
n=3 impar
$p_0 = 8,\ \ p_1 = 1 * 8 + 1 = 9,\ \ p_2 = 1 * 9 + 8 = 17,\ \ p_3 = 43$
$q_0 = 1,\ \ q_1 = 1,\ \ q_2 = 1 * 1 + 1 = 2,\ \ q_3 = 5 \Rightarrow$

$$43(2) + 5(-17) = 86 - 85 = 1$$

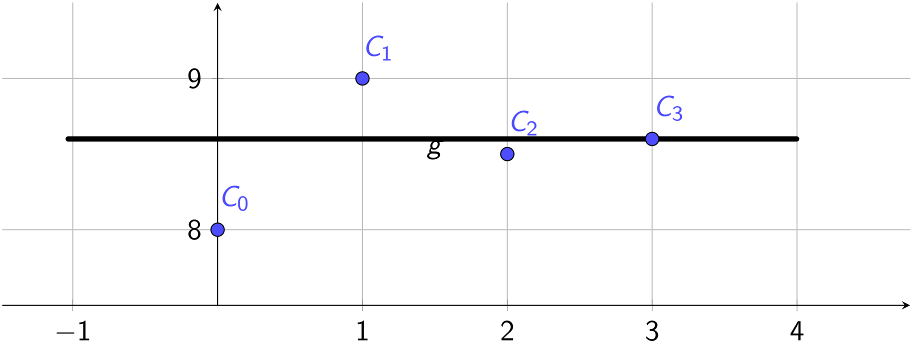
$$43(500) + 5(-4250) = 43(2 * 250) + 5(-17 * 250) = 250$$

$$172(500) + 20(-4250) = 1000$$

Contenido

coportamiento de los convergentes

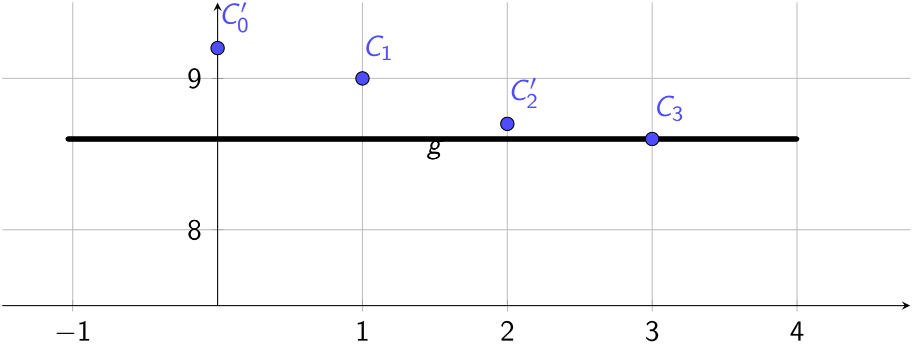
$$C_0 = 8/1 = 8,\ \ C_1 = 9/1 = 9, C_2 = 17/2 = 8.5,\ \ C_3 = 43/5 = 8.6$$



comportamiento de los convergentes

$$C_0' = 2 * 43/5 - 8/1 = 9.2, \;\; C_1 = 9/1 = 9,$$

$$C_2' = 2 * 43/5 - 17/2 = 8.7, \;\; C_3 = 43/5 = 8.6$$



ejemplo 493/284

$$
\begin{aligned}
493 &= 1 \cdot 284 & +209 \\
284 &= 1 \cdot 209 & +75 \\
209 &= 2 \cdot 75 & +59 \\
75 &= 1 \cdot 59 & +16 \\
59 &= 3 \cdot 16 & +11 \\
16 &= 1 \cdot 11 & +5 \\
11 &= 2 * 5 & +1 \\
5 &= 5 \cdot 1 &
\end{aligned}
$$

$\Rightarrow \frac{493}{284} = [1; 1, 2, 1, 3, 1, 2, 5]$

$$p_0 = 1, p_1 = 2, p_2 = 5, p_3 = 7, p_4 = 26, p_5 = 33, p_6 = 92, p_7 = 493$$

$$q_0 = 1, q_1 = 1, q_2 = 3, q_3 = 4, q_4 = 15, q_5 = 19, q_6 = 53, q_7 = 284$$

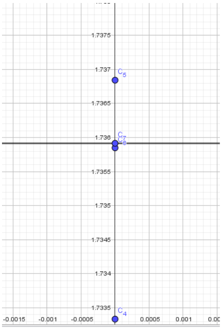
ejemplo

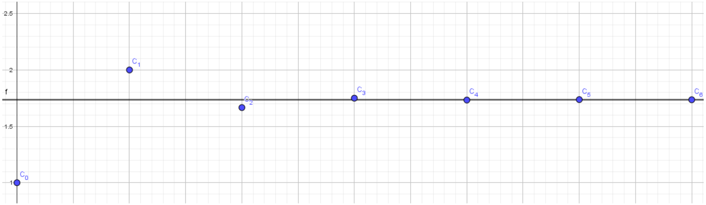
$$C_0 = 1, C_1 = 2, C_2 = \frac{5}{3}, C_3 = \frac{7}{4}, C_4 = \frac{26}{15}, C_5 = \frac{33}{19}, C_6 = \frac{92}{53}, C_7 = \frac{493}{284}$$

$$C_0 = 1, c_2 = 1.6, C_4 = 1.733, C_6 = 1.7358, C_7 = 1.7359, C_5 = 1.736,$$

$$C_3 = 1.75, C_1 = 2$$

$$C_0' = .7, C_1' = .2, C_2' = .06, C_3' = .01, C_4' = .002, C_5' = .0009, C_6' = .00006$$





Lemma $q_{k-1} < q_k, \forall k \in \mathbb{N} \cap [2, n]$

Demostración.

Sabemos por definicion que

$$q_k = a_k q_{k-1} + q_{k-2}$$

donde $a_k, q_{k-2} \in \mathbb{Z}^+ \Rightarrow q_{k-2} > 0, a_k \geq 1 \Rightarrow$

$$q_k = a_k q_{k-1} + q_{k-2} > a_k q_{k-1} + 0 \geq 1 q_{k-1} = q_{k-1}$$

$\square$

$q_1 = a_1 \geq 1 = q_0$

Teorema 15.4 $C_{2k} < C_{2k+2} < C_{2k'+3} < C_{2k'+1}, \forall k, k' \in \mathbb{N}$

$$C_0 < C_2 < \cdots < C_{2\lfloor n/2 \rfloor} < C_{2\lceil n/2 \rceil - 1} < \cdots < C_1$$

Demostración.

$$C_{k+2} - C_k = (C_{k+2} - C_{k+1}) + (C_{k+1} - C_k) = \frac{(-1)^{k+1}}{q_{k+2}q_{k+1}} + \frac{(-1)^k}{q_{k+1}q_k} =$$

$$= \frac{(-1)^k(q_{k+2} - q_k)}{q_{k+2}q_{k+1}q_k}$$

por el lema anterior $q_{k+2} > q_k$ y por definicion $q_i > 0 \Rightarrow$
Si k es par $(-1)^k > 0 \Rightarrow C_{k+2} > C_k$
y si es impar $(-1)^k < 0 \Rightarrow C_{k+2} < C_k$
Ahora como $C_{k+1} - C_k = \frac{(-1)^k}{q_{k+1}q_k}$, analogamente,
Si k es par $C_{k+1} - C_K > 0$, $C_{k+1} > C_k$ impar>par
y si k es impar $C_{k+1} - C_K < 0$, $C_{k+1} < C_k$ par<impar
de culquier forma, el par es el menor

Contenido

Referencias

David M. Burton
ELEMENTARY NUMBER THEORY .
Seven Edition