Coordenadas polares y el volumen de la esfera unitaria

Seminario de Teoria de la Medida

Alejandro Pallais Garcia

Universidad del Valle de Guatemala

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Contenido

Contenido

Teorema 16.4 (transformada de Jacobi)

Sean $U, V \in \tau_{\mathbb{R}^n}$, $\Phi : U \to V$ un C^1-difeomorfismo y $\lambda_W = \lambda^n(\cdot \cap W) \Rightarrow$

$$\lambda_V(X) = \Phi(|detD\Phi(X)|\lambda_U)$$

equivalentemente

$$\int_V f(y)dy = \int_U f(\Phi(x))|detD\Phi(X)|dx, \forall f \in \mathbb{M}^+(\mathbb{B}(V))$$

particularmente

$$f \in \mathbb{L}^1(\lambda_V) \Leftrightarrow |DetD\Phi| f \circ \Phi \in \mathbb{L}^1(\lambda_U)$$

Corolario 16.10

Sean $B \in \overline{\mathbb{B}}(\mathbb{R}^n)$, $U = B^o$, $U' \in \tau_{\mathbb{R}^n} \ni B \subseteq U'$ y $\Phi : U' \to \mathbb{R}^n$ una funcion lipchitz continua, Si $B/U \in N(\overline{\lambda^n})$ y $\Phi_U : U \to \Phi(U)$ un C^1-diffeomorphismo $\Rightarrow$

$$\int_{\Phi(B)} f(y)\overline{\lambda^n}(dy) = \int_B f(\Phi(x))|detD\Phi(X)|\overline{\lambda^n}(dx), \forall f \in \mathbb{M}^+(\mathbb{B}(V))$$

particularmente

$$f \in \mathbb{L}^1(\overline{\lambda^n}, \Phi(B)) \Leftrightarrow |DetD\Phi| f \circ \Phi \in \mathbb{L}^1(\overline{\lambda^n}, B)$$

Contenido

Definicion de coordenadas polares en $\mathbb{R}^2$

consideremos el mapeo

$\Psi : (0,\infty)\times(-\pi,\pi) \to \mathbb{R}^2/((-\infty,0]\times\{0\}) \ni \Psi(r,\theta) = (r\cos(\theta), r\sin(\theta))$

Facilmente podemos ver que Ψ es un C^1-difeomorfismo y su jacobiano es

$$detD\Phi(X) = det\left(\frac{\partial\Psi(r,\theta)}{\partial(r,\theta)}\right) = det\begin{pmatrix}(\partial/\partial r)(r\cos(\theta)) & (\partial/\partial\theta)(r\cos(\theta))\\ (\partial/\partial r)(r\sin(\theta)) & (\partial/\partial\theta)(r\sin(\theta))\end{pmatrix} =$$

$$= det\begin{pmatrix}\cos(\theta) & -r\sin(\theta)\\ \sin(\theta) & r\cos(\theta)\end{pmatrix} = r\cos^2(\theta) + r\sin^2(\theta) = r$$

$\Rightarrow$ por el corolario 16.10 y teorema de Fubini

$$\int_{\mathbb{R}^2} f(x,y)d(x,y) = \int_{(0,\infty)\times(-\pi,\pi)} f(\Phi(x))|detD\Phi(X)|d(r,\theta) =$$

$$= \int_{(0,\infty)\times(-\pi,\pi)} rf(r\cos(\theta), r\sin(\theta))d(r,\theta) =$$

$$= \int_{(0,\infty)}\int_{(-\pi,\pi)} rf(r\cos(\theta), r\sin(\theta))d(r)d(\theta), \forall f \in \mathbb{M}^+(\mathbb{B}(V))$$

Ejemplo

$$\int_{\mathbb{R}} e^{-x^2} d\lambda^1(x) = \sqrt{\left(\int_{\mathbb{R}} e^{-x^2} d\lambda^1(x)\right)^2} = \sqrt{\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-x^2} e^{-y^2} dxdy} \overset{\text{Tonelli}}{=}$$

$$= \sqrt{\iint_{\mathbb{R}^2} e^{-(x^2+y^2)} d(x,y)} \overset{\text{polares}}{=} \sqrt{\int_{(0,\infty)}\int_{(-\pi,\pi)} re^{-(r^2\cos^2(\theta)+r^2\sin^2(\theta))} d\theta dr} =$$

$$= \sqrt{\int_{(0,\infty)}\int_{(-\pi,\pi)} re^{-r^2} d\theta dr} = \sqrt{\int_{(0,\infty)} re^{-r^2}\int_{(-\pi,\pi)} 1 d\theta dr} =$$

$$= \sqrt{\int_{(0,\infty)} re^{-r^2}\lambda^1(-\pi,\pi) dr} = \sqrt{2\pi\int_0^{\infty} re^{-r^2} dr} = \sqrt{2\pi[-\frac{1}{2}e^{-r^2}]_0^{\infty}} = \sqrt{\pi}$$

Contenido

Definicion de coordenadas polares en $\mathbb{R}^n$

consideremos el mapeo

$$\Psi : (0,\infty) \times (0,\pi)^{n-2} \times (-\pi,\pi) \to \mathbb{R}^n / \{X : x_n = 0, x_{n-1} \leq 0\}$$

$\ni X = (x_i)_{i=1}^{n} \in \mathbb{R}^n, \theta = (\theta_i)_{i=0}^{n-1} \in (0,\infty) \times (0,\pi)^{n-2} \times (-\pi,\pi), \theta_0 = r \ni$ $\Psi(\theta) = X \Rightarrow x_n = r \prod_{i=1}^{n-1} \sin(\theta_i)$ y

$$x_i = r\cos(\theta_i) \prod_{j=1}^{i-1} \sin(\theta_j), \forall i < n$$

Lemma 16.18 Ψ es un C^1-difeomorfismo y su jacobiano es

$$detD\Psi(\theta) = r^{n-1} \prod_{i=1}^{n-2} \sin^{n-1-i}(\theta_i)$$

ejemplos

n=2

$$\Psi : (0,\infty) \times (0,\pi)^{2-2} \times (-\pi,\pi) \to \mathbb{R}^2/\{X : x_2 = 0, x_{2-1} \leq 0\}$$

$$\Psi : (0,\infty) \times (-\pi,\pi) \to \mathbb{R}^2/\{(x,y) : y = 0, x \leq 0\}$$

$$detD\Psi(r,\theta) = r^{2-1} \prod_{i=1}^{2-2} \sin^{2-1-i}(\theta_i) = r$$

n=3

$$\Psi : (0,\infty) \times (0,\pi)^{3-2} \times (-\pi,\pi) \to \mathbb{R}^3/\{X : x_3 = 0, x_{3-1} \leq 0\}$$

$$\Psi : (0,\infty) \times (0,\pi) \times (-\pi,\pi) \to \mathbb{R}^3/\{(x,y,z) : z = 0, y \leq 0\}$$

$$detD\Psi(r,\theta_1,\theta_2) = r^{3-1} \prod_{i=1}^{3-2} \sin^{3-1-i}(\theta_i) = r^2 \sin(\theta_1)$$

Contenido

16.19

$u(x) \in \mathbb{L}^1(dx) \Leftrightarrow |j(r,\theta)|u(\Psi(\theta)) \in \mathbb{L}^1(d\theta)$ en este caso

$$\int_{\mathbb{R}^n} u(x)dx = \int_0^\infty \overbrace{\int_0^\pi \cdots \int_0^\pi}^{n-2} \int_{-\pi}^{\pi} u(\Psi(\theta))|j(\theta)|d\theta$$

Si $\mathbb{S}^{n-1}$ es la esfera de dimencion n-1 en $\mathbb{R}^n \Rightarrow$

$$\sigma_{n-1}(\Gamma) = \overbrace{\int_0^\pi \cdots \int_0^\pi}^{n-2} \int_{-\pi}^{\pi} 1_\Gamma(\Psi(\theta/\theta_n, r=1))|j(\theta)|d\theta/\theta_n$$

es la medida superficial canonica sobre $\mathbb{S}^{n-1}$ en este caso

$$\int_{\mathbb{R}^n} u(x)dx = \int_0^\infty \overbrace{\int_0^\pi \cdots \int_0^\pi}^{n-2} \int_{-\pi}^{\pi} u(\Psi(\theta))|j(\theta)|d\theta = \int_0^\infty \int_{\mathbb{S}^{n-1}} r^{n-1}u(rs)\sigma_{n-}$$

volumen-superficie

veamos que el volumen de la esfera esta dado por

$$V = \lambda^n(\mathbb{S}^{n-1}) = \int_{\mathbb{R}^n} u(x)dx = \int_0^1 \overbrace{\int_0^\pi \cdots \int_0^\pi}^{n-2} \int_{-\pi}^{\pi} |J(\theta)| d\theta =$$

$$= \int_0^1 r^{n-1} \overbrace{\int_0^\pi \cdots \int_0^\pi}^{n-2} \int_{-\pi}^{\pi} |J(\theta, r=1)| d\theta = \int_0^1 r^{n-1} \sigma_{n-1}(\mathbb{S}^{n-1}) dr =$$

$$= \sigma_{n-1}(\mathbb{S}^{n-1}) \int_0^1 r^{n-1} dr = \frac{1}{n} \sigma_{n-1}(\mathbb{S}^{n-1})$$

16.20

si $f(x) = \phi(|x|)$ es una funcion de rotacion simetrica

$$\int_{\mathbb{R}^n} f(x)dx = \int_0^\infty \int_{\mathbb{S}^{n-1}} r^{n-1} f(rs)\sigma_{n-1} dsdr = \int_0^\infty r^{n-1} \int_{\mathbb{S}^{n-1}} \sigma_{n-1} f(rs) dsdr =$$

$$= \int_0^\infty r^{n-1}\sigma_{n-1}(\mathbb{S}\phi(r)dr = \sigma_{n-1}(\mathbb{S}) \int_0^\infty r^{n-1}\phi(r)dr = nV \int_0^\infty r^{n-1}\phi(r)dr$$

Contenido

Demostracion

$$(\sqrt{\pi})^n = \left(\int_{\mathbb{R}} e^{-x^2} d\lambda^1(x)\right)^n = \overbrace{\int_{\mathbb{R}} \cdots \int_{\mathbb{R}}}^{n} e^{-\sum_{i=1}^n x_i^2} \lambda^1(dx) =$$

$$= \int_{\mathbb{R}^n} e^{-\sum_{i=1}^n x_i^2} \lambda^1(dx) = nV \int_0^\infty r^{n-1} e^{-r^2} dr = \frac{1}{2} nV \int_0^\infty s^{(1/2)n-1} e^{-s} ds =$$

$$= \frac{n}{2} V\Gamma(\frac{n}{2}) = V\Gamma(\frac{n}{2}+1)$$

$\Rightarrow V = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$

y el area superficial es $\sigma_{n-1}(\mathbb{S}) = nV = 2\frac{\pi^{n/2}}{\Gamma(\frac{n}{2})}$

Contenido

Referencias

Rene L. Schilling
Measures, Integrals and Martingales.
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