Teorema (de Convergencia Dominada de Lebesgue):

Sea $(X, \mathcal{A}, \mu)$ espacio de medida y $\{u_n\}_{n\geq 1} \subseteq L^1(\mu)$ una sequencia de funciones integrables tales que $|u_n(x)| \leq w(x)$, $\forall x \in X$, $\forall n \in \mathbb{N}$, para alguna $w \in L^1(\mu)$, y suponga que $u = \lim_{n\to\infty} u_n$. Entonces

i) $\lim_{n\to\infty} \int |u_n - u| \, d\mu = 0$

ii) $u \in L^1(\mu)$ y $\int u \, d\mu = \int \lim_{n\to\infty} u_n \, d\mu = \lim_{n\to\infty} \int u_n \, d\mu$.

Prueba: De la hipótesis $|u_n| \leq w$, $\forall n$ $\Rightarrow$ $\lim_{n\to\infty} |u_n| = |u| \leq w$.

Entonces $\int |u| \, d\mu \leq \int w \, d\mu < \infty$ $\quad (w \in L^1)$

$\Rightarrow |u| \in L^1(\mu) \Rightarrow u \in L^1(\mu)$.

$(i \Rightarrow ii)$ $\left| \int u_n d\mu - \int u d\mu \right| = \left| \int (u_n - u) d\mu \right| \leq \int |u_n - u| d\mu$

$$\lim_{n\to\infty} \left| \int u_n d\mu - \int u d\mu \right| \leq \lim_{n\to\infty} \int |u_n - u| d\mu = 0$$

$$\Rightarrow \left| \lim_{n\to\infty} \int u_n d\mu - \int u d\mu \right| = 0$$

$$\Rightarrow \underline{\lim_{n\to\infty} \int u_n d\mu = \int u d\mu = \int \lim_{n\to\infty} u_n d\mu} \quad (ii)$$

th

(i) De la desigualdad triangular $|u_n - u| \leq |u_n| + |u| \leq w + w = 2w$

$\Rightarrow \underline{2w - |u_n - u| \geq 0}$, $\forall n \in \mathbb{N}$. Por el Lema de Fatou

$$\int \cancel{2w} d\mu = \int \liminf_{n\to\infty} (2w - |u_n - u|) d\mu \leq \liminf_{n\to\infty} \int (2w - |u_n - u|) d\mu$$

$$\leq \int \cancel{2w} d\mu - \limsup_{n\to\infty} \int |u_n - u| d\mu$$

$$\Rightarrow -\limsup \int |u_n - u| \, d\mu \geq 0 \Rightarrow \limsup \int |u_n - u| \, d\mu \leq 0$$

$$\Rightarrow 0 \leq \liminf_{n\to\infty} \int |u_n - u| \, d\mu \leq \limsup_{n\to\infty} \int |u_n - u| \, d\mu \leq 0$$

$$\Rightarrow \liminf_{n\to\infty} \int |u_n - u| \, d\mu = 0 = \limsup_{n\to\infty} \int |u_n - u| \, d\mu$$

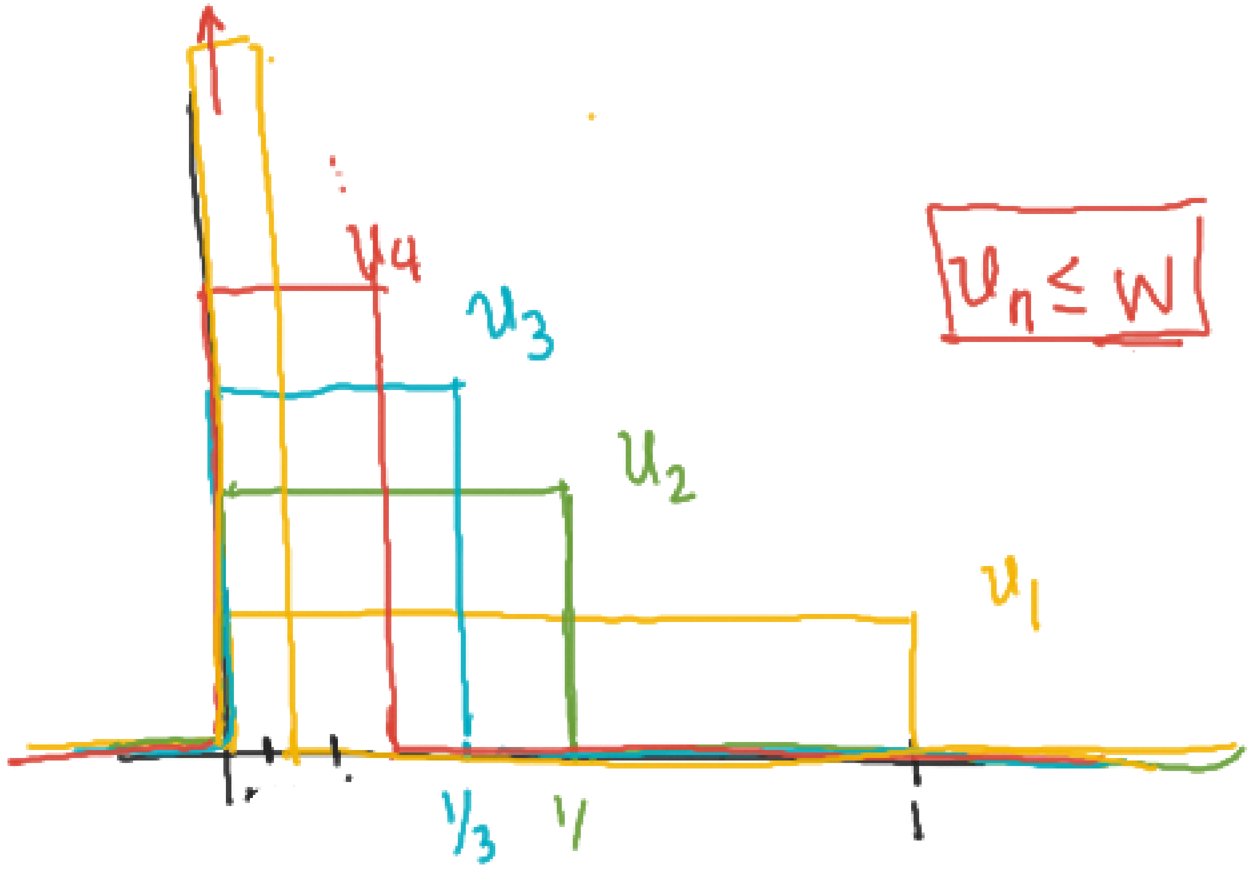
$$\Rightarrow \lim_{n\to\infty} \int |u_n - u| \, d\mu = 0 \quad (i).$$

$\square$

Obs! En el T. de Conv. Dominada, la hipótesis de u_n ser dominada uniformemente por $w \in \mathcal{L}^1(\mu)$ es esencial.

Ej: $(X, \mathcal{A}, \mu) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda^1)$. Definimos la secuencia $u_n : \mathbb{R} \to \overline{\mathbb{R}}$ por

$$u_n(x) = n \cdot \mathbb{1}_{[0, 1/n]}(x) = \begin{cases} n; & \text{si } 0 \leq x \leq 1/n \\ 0; & \text{otro caso} \end{cases}$$



$\boxed{u_n \leq w}$

Las u_n son funciones simples.

$\Rightarrow \quad u_n \in \mathcal{M}^+_{\mathbb{R}}(\lambda^1)$.

$u_n = n\mathbb{1}_{[0,1/n]} \xrightarrow[n\to\infty]{} 0 \quad \underline{\lambda^1 - \text{c.t.p.}}$

Por otro lado
$$\int_{\mathbb{R}} u_n \, d\lambda^1 = \int_{\mathbb{R}} n \cdot \mathbb{1}_{[0,1/n]} \, d\lambda^1 = \int_{[0,1/n]} n \, d\lambda^1$$
$$= n\int_{[0,1/n]} d\lambda^1 = n\,\lambda^1(0,1/n) = n \cdot 1/n = 1, \ \forall n$$

$$\Rightarrow \underline{\lim_{n\to\infty} \int u_n \, d\lambda^1 = \lim_{n\to\infty} 1 = 1} \quad \text{y} \quad \underline{\int \lim u_n \, d\lambda^1 = \int 0 \, d\lambda^1 = 0}.$$

$$\lim_{n\to\infty} \int u_n \neq \int \lim_{n\to\infty} u_n$$

Riemann v/s. Lebesgue: $(X, \mathcal{A}, \mu) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda^1)$.

- Rieman tiene limitantes (inadecuada para tratar ciertas funciones)
- motivación de J. Lebesgue $\rightsquigarrow$ resolver esta limitante
- para una familia amplia de funciones $u:[a,b]\to\mathbb{R}$

$$\int u \, d\lambda^1 = \int_a^b u(x)\,dx$$

Riemann: $\boxed{P=\{a=t_0<t_1<t_2<\cdots<t_n=b\}}$ partición de $[a,b]$

$$s(P,u)=\sum_{i=1}^{n} m_i(t_i-t_{i-1}), \quad S(P,u)=\sum_{i=1}^{n} M_i(t_i-t_{i-1})$$

donde

$$m_i = \inf_{t_{i-1}\le x\le t_i} u(x) \quad \text{y} \quad M_i = \sup_{t_{i-1}\le x\le t_i} u(x)$$

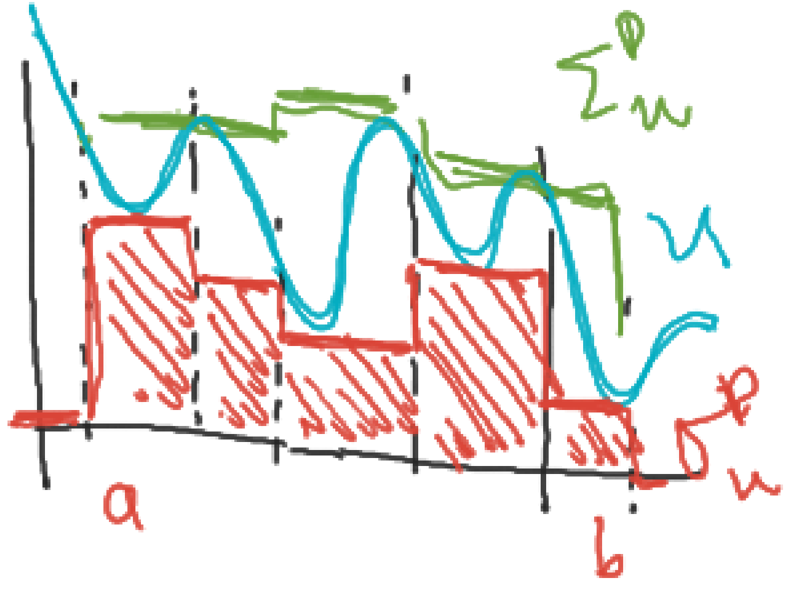
$$\underline{\int} u\,dx = \sup_P s(P,u) \quad \text{y} \quad \overline{\int} u\,dx = \inf_P S(P,u).$$

u es Riemann-integrable $\iff \underline{\int} u\,dx = \overline{\int} u\,dx.$

Obs • Si $Q \supseteq P \Rightarrow s(P,u) \le s(Q,u) \le S(Q,u) \le S(P,u).$

• A $s(P,u)$ y $S(P,u)$ le corresponden funciones simples

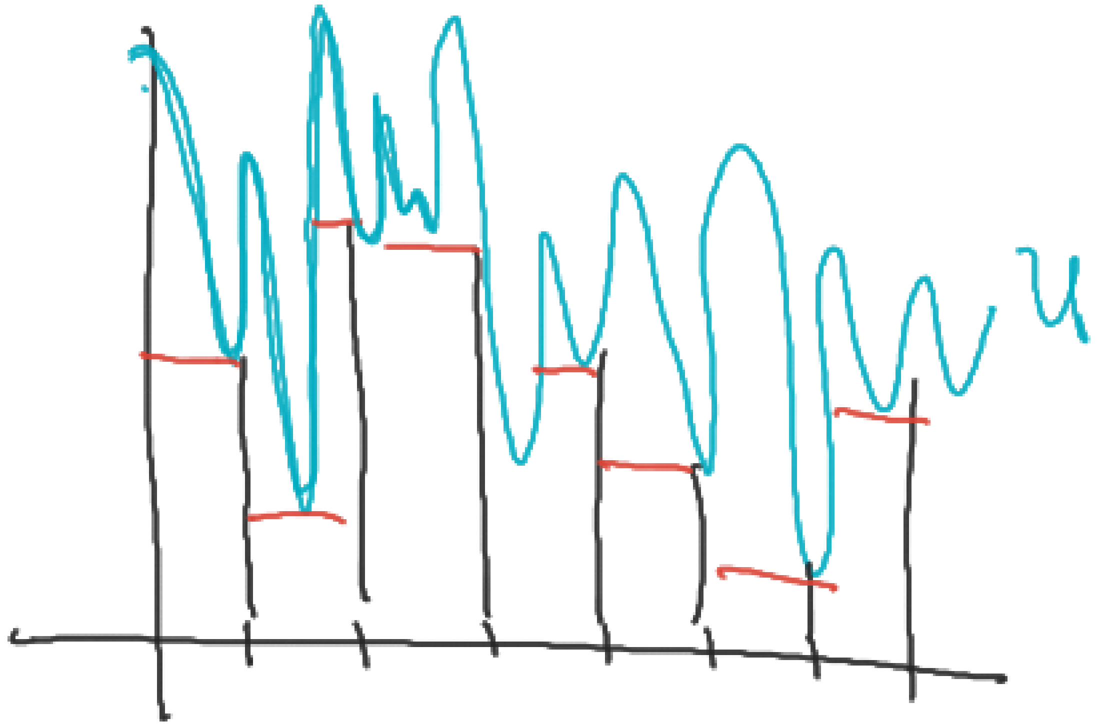
$$\sigma_u^P(x) = \sum_{i=1}^n m_i \mathbb{1}_{[t_{i-1},t_i]}(x) \quad \text{y} \quad \Sigma_u^P(x) = \sum_{i=1}^n M_i \mathbb{1}_{[t_{i-1},t_i]}(x)$$



$$\Rightarrow \sigma_u^P \le u \le \Sigma_u^P, \quad \forall P \text{ partición.}$$

$$\int_a^b \sigma_u^P\,dx = s(P,u).$$

$$\int_a^b \Sigma_u^P\,dx = S(P,u).$$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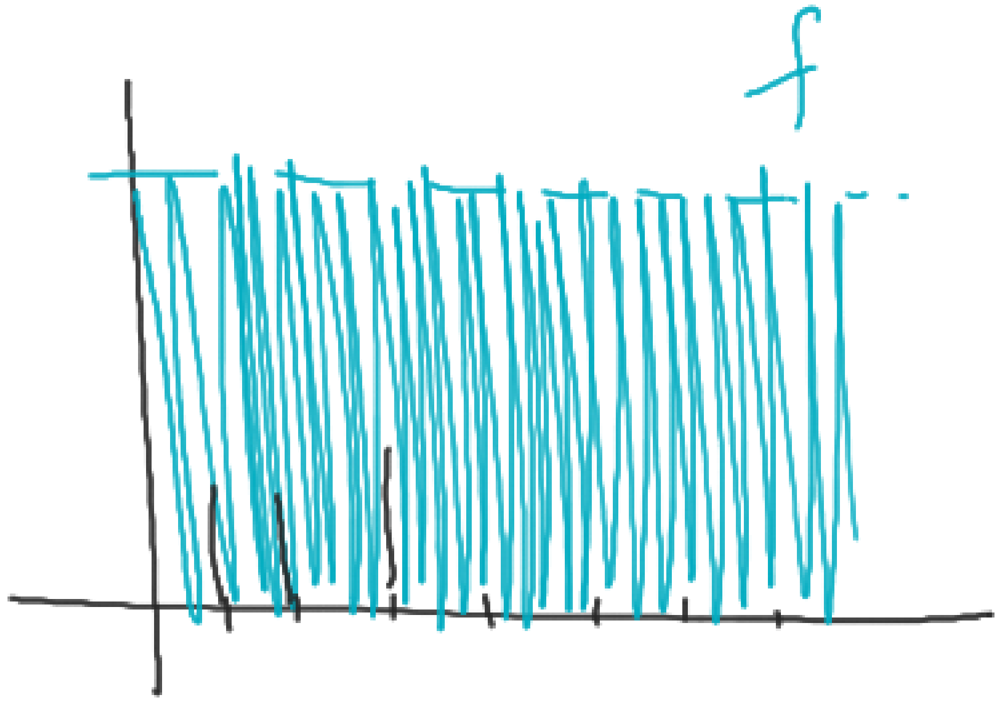
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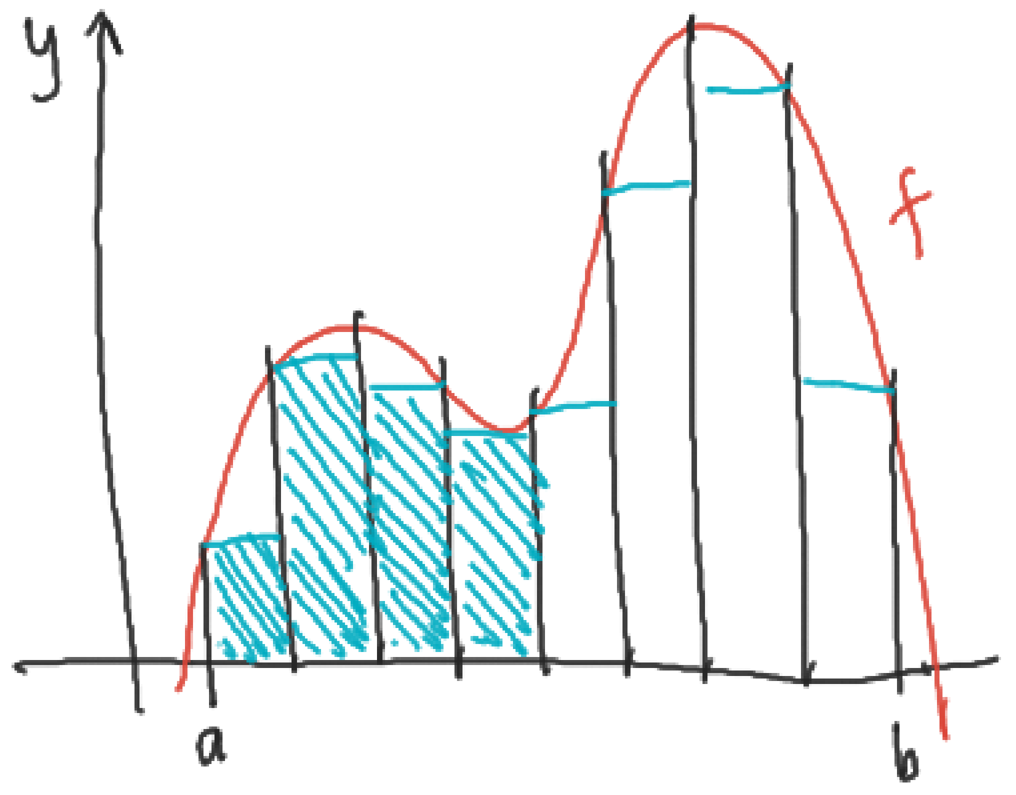
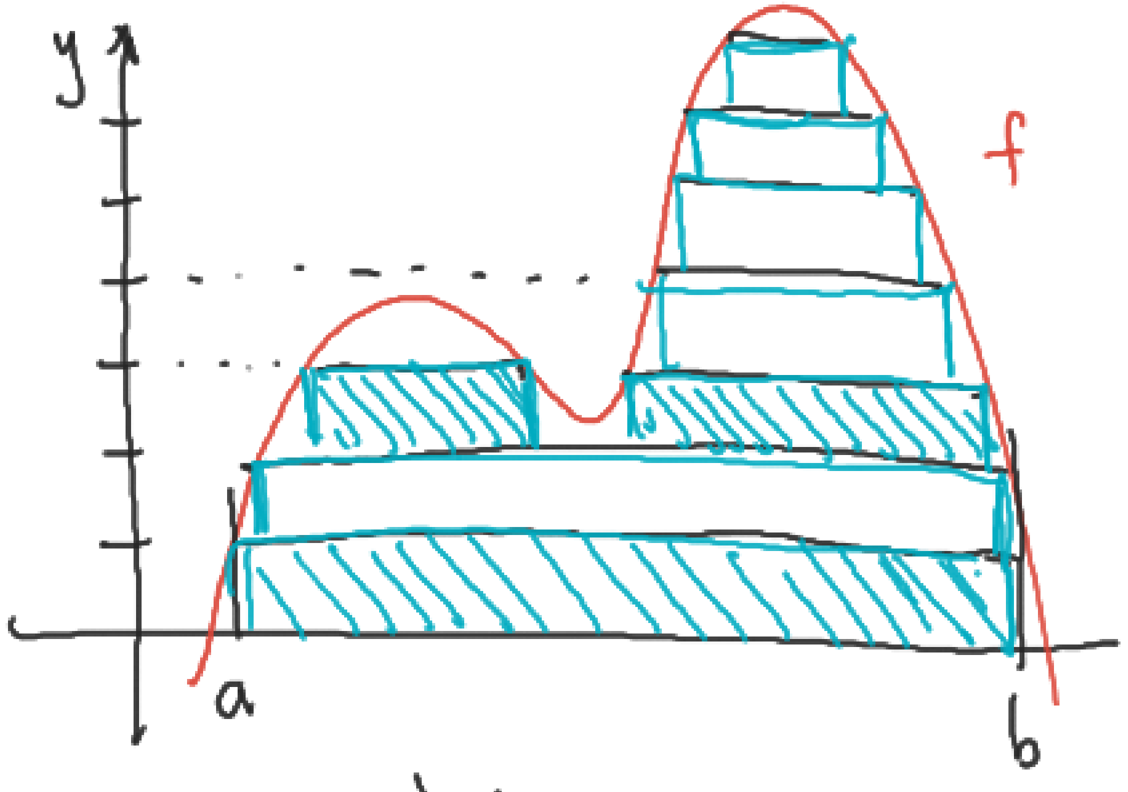


Riemann



Lebesgue

Teorema: Sea $u:[a,b]\longrightarrow\mathbb{R}$ mesurable y Riemann integrable.

Entonces, $u\in L^1(\lambda^1)$ y

$$\int_{[a,b]} u\,d\lambda^1 = \int_a^b u(x)\,dx\,,$$

Prueba: Como u es Riemann integrable $\Rightarrow \exists$ secuencia de particiones

$$P_{(1)} \subseteq P_{(2)} \subseteq P_{(3)} \subseteq \ldots \subseteq P_{(k)} \subseteq \ldots$$

tales que

$$\lim_{k\to\infty} s(P_{(k)}, u) = \underline{\int} u \quad \text{y} \quad \lim_{k\to\infty} S(P_{(k)}, u) = \overline{\int} u .$$

$$\text{y} \quad \underline{\int} u = \overline{\int} u ,$$

Las secuencias de funciones simples $\sigma_k = \sigma_u^{P_{(k)}}$ y $\Sigma_k = \Sigma_u^{P_{(k)}}$ son monótonas

$$\sigma_1 \le \sigma_2 \le \sigma_3 \le \ldots\ldots \le \Sigma_3 \le \Sigma_2 \le \Sigma_1$$

y convergen monótonamente a

$$\sigma_k \nearrow \sigma_u \qquad \Sigma_k \searrow \Sigma_u$$

$\sigma_u \le u \le \Sigma_u$.

Por Convergencia Monótona $\left(\sigma_k, \Sigma_k \in L^1(\lambda)\right)$

$$\underline{\int} \sigma_u = \lim_{k\to\infty} s(P_{(k)}, u) = \lim_{k\to\infty} \int \sigma_k \, d\lambda^1 = \int \sigma_u \, d\lambda^1$$

$$\overline{\int} \Sigma_u = \lim_{k\to\infty} S(P_{(k)}, u) = \lim_{k\to\infty} \int \Sigma_k \, d\lambda^1 = \int \Sigma_u \, d\lambda^1$$

$$\Rightarrow \int_{[a,b]} \underbrace{(\Sigma_u - \sigma_u)}_{\geq 0} d\lambda = \int \Sigma_u \, d\lambda - \int \sigma_u \, d\lambda = \overline{\int} \Sigma_u - \underline{\int} \sigma_u$$
$$= \overline{\int} u - \underline{\int} u = 0$$

$\Rightarrow \Sigma_u - \sigma_u = 0 \;\; \lambda^1\text{-c.t.p.} \quad \Rightarrow \Sigma_u \equiv \sigma_u \;\; \text{c.t.p.} \quad (\sigma_u \leq u \leq \Sigma_u)$

$\Rightarrow \{u \neq \sigma_u\} \cup \{u \neq \Sigma_u\} \subseteq N_{\lambda^1} \Rightarrow \underset{\substack{\uparrow \\ L^1}}{u = \sigma_u} \;\; \text{c.t.p.}$

$\Rightarrow u \in L^1(\lambda^1).$

Como $u = \sigma_u$ c.t.p. y $u = \Sigma_u$ c.t.p.

$$\Rightarrow \int u\,d\lambda = \int \sigma_u\,d\lambda = \underline{\int} u \quad \text{y} \quad \int u\,d\lambda = \int \Sigma_u\,d\lambda = \overline{\int} u$$

$$\Rightarrow \quad \boxed{\int u\,d\lambda' = \int_a^b u(x)\,dx.} \quad \square$$