# Outline of a history of differential geometry (*)

## I

### 1. — *The Time before Leibniz.*

It is difficult to talk of differential geometry before LEIBNIZ. There were many applications of infinitesimals to geometry before, but almost without exception they were quadratures and rectifications of curves, cubatures and quadratures of surfaces or solids, and studies of special curves, subjects we now exclude from differential geometry proper, except as occasional illustrations. Only a few topics have an immediate bearing of our subject.

Among them we have to mention the investigation of the nature of tangency found in EUCLID's " Elements " (last part of 4th century, B.C.), discussed only in the case of a circle. EUCLID explains, in Book III, as a " definition " that a straight line is tangent to a circle if it meets the circle and does not intersect it after being continued. Also in Proposition 16 of the same book a property of tangency is explained. The tangent to a circle, it is said here, will fall outside the circle and no other line will fall in the space between this straight line and the circumference. This property of the tangent was taken up again later and generalized by LAGRANGE, when he developed his theory of contact (*Théorie des fonctions analytiques*, Seconde partie I, 5). EUCLID himself tries to describe the nature of contact more in detail in the same proposition, in which he states that " the angle of the semicircle is larger than any rectilinear acute angle, the remaining angle smaller." This suggestion of extending the notion of

(*) This outline was given in a series of ten lectures at the Massachusetts Institute of Technology during fall and winter of 1931-32.

angle to so-called horned angles, though having been the subject of long discussions, has not been carried far in differential geometry. (1)

Similar ideas can be found in the works of ARCHIMEDES and APOLLONIUS. APOLLONIUS (3d century B.C.) makes full use of the normal to a plane curve in his theory of conic sections. He goes further; he finds that the normals to a conic section have an envelope, and he determines this envelope in the three cases of an ellipse, parabola and hyperbola. (2) He comes so closely to the conception of curvature that KEPLER, in a book on optics, could talk of the circle of curvature at a point of a parabola, as if it were well-known to all his readers. (3)

In the same treatise on conic sections (in book II) we find the asymptotes to a hyperbola. The technical name for these lines is also due to APOLLONIUS (ἀσυμπτώτη).

ARCHIMEDES (287-212 B.C.) occasionally discusses subject matter relevant to differential geometry, as in the beginning of his books on the sphere and the cylinder, where he defines the straight line as the shortest distance between two points in the plane. He states in this work the definition of curves " concave in the same direction " and arrives at the statement that if two plane curve segments with the same endpoints are concave in the same direction, the curve lying between the straight line connecting the two points and the other curve is shorter than this other curve. He establishes similar theorems for surfaces. This paper includes, for instance, the theorem that when a convex plane curve lies inside another convex plane curve, its circumference is the shorter. The curves also may be partly or entirely composed of line segments, or may partly coincide. (4)

The problem of isoperimetrical figures belongs also to antiquity, and is now included in that part of differential geometry which utilizes calculus of variations. POLYBIUS, historiograph of the Punic

(1) Comp. F. KLEIN, Elementarmathematik vom höheren Standpunkt aus. (Berlin, SPRINGER) 1925, p. 222 sequ. See E. KASNER, *Bulletin Amer. Math. Soc.* (2) 17 (1910-11) p 393.

(2) APOLLONIUS OF PERGA, ed. TH. HEATH (Cambridge, 1892), p. 160-179.

(3) J. KEPLER, " Paralipomena in Vitellionem, " Ch. III, Theor. XIX, Werke II, p. 175. (*Ostwald's Klass.* 198, p. 54).

(4) See e. g. ARCHIMEDES, French translation of VER EECKE, p. 4-6.

wars, remarks that most people measure the size of towns or camps by their perimeter, and that they can hardly believe that Sparta, with a circumference of 48 stadia, is twice as large as Megalopolis with a circumference of 50 stadia. (5) The mathematics of this problem seems to go back to the time before ARISTOTLE. A series of theorems is found in a paper by ZENODORUS (about 150 B.C.), who stated that the circle is larger than all plane figures of the same circumference, and the sphere larger than all solid figures of the same area. Exact proofs, as a matter of fact, date back only to the 19th century.

Then there is the problem of mapping the earth on a plane, a problem which offered itself to those geographers of Antiquity who recognized the earth as a sphere. The principal contribution is due to PTOLEMY (150 A.D.), though we may readily believe that his ideas were those of HIPPARCH, who lived three centuries earlier. In PTOLEMY's *Geography*, Chapter 24, we find what we now call the stereographic projection. He takes the equator as plane of mapping, and he not only explains the projection, but also shows its conformal character. He also modifies the projection by mapping the figure on a cone tangent to the sphere. This allows a good representation of that part of the earth known to PTOLEMY. The " map of the world according to PTOLEMY," reproduced in many textbooks, is drawn in this projection. (6)

We cannot deal here with the reasons for the slow progress of differential geometry in Antiquity, as it is only one aspect of the much more general problem why antiquity did not advance beyond ARCHIMEDES in the calculus of infinitesimals. We remark that it has to do only, in a general way, with the fact that the economic system on which the Mediterranean culture was based already began to decay in the last centuries of the Roman republic.

Many new methods of mapping a sphere on a plane were invented in the sixteenth century, when the great discoveries gra-

(5) POLYBIUS lived from 201-119 B.C. The place is from his Books IX, 21. Our information on isoperimetry is taken from W. SCHMIDT, " Geschichte der Isoperimetrie im Altertume," *Bibliotheca mathematica* (3)2 (1901), p. 5-8. See also M. CANTOR, Vorlesungen I, 3[d] ed., p. 357.

(6) Map projections also in PTOLEMY's " Planisphaerum " and " Analemma." The name " stereographic projection " is due to F. D'AIGUILLON (1566-1617), a Belgian Jesuit, who has Monge, central, and stereographic projection in his " Optics " (1613). (He calls the first two orthographic and scenographic).

dually widened the knowledge of the terrestrial sphere. In 1540, GEMMA FRISIUS, professor at Louvain, again used the stereographic projection. Of greater importance is the work of GERHARD KRÄMER, Latin MERCATOR (1512-1594), a Flemish cartographer who lived a good part of his life at Duisburg. He used many map projections, of which one carries his name, because he used it for the first time in the famous map of the world of 1569. This method, the only one invented by the great cartographer, projects meridians and parallels into straight lines. MERCATOR knew the properties of his map very well, for instance its conformity, and the meaning of the straight lines. He discriminates between " plaga " and " directio," the " plaga " being the shortest connection between two points on the earth, the " directio " the shortest distance on the map. This " directio," in the words of MERCATOR, is not straight, but " oblique curvatur." For large distance and high latitude there is considerable difference between " plaga " and " directio". (7)

There was a considerable literature on MERCATOR projection in the next decades, and connected with it we find a discussion of the " directio." NUNES (as early as 1544, in print 1573), STEVIN, SNELLIUS, WALLIS, LEIBNIZ contributed. The name " loxodrome " is due to SNELLIUS' *Typhys Batavus* (1624).

Earlier than FRISIUS and MERCATOR, J. WERNER suggested, after JOH. STABER, a projection (1514), which conserves areas. It was used in 1531 by O. FINAEUS for a map of the world, and in 1538 by MERCATOR. This map, with its curious heart-like shape, is seldom used. (8)

The many new investigations on curves and on infinitesimals connected with the names of KEPLER, DESCARTES, FERMAT, CAVALIERI and others are mostly of too special, or too general a nature to find discussion here. A point of inflexion was first discussed by DE SLUSE (1668) and FERMAT (1679). (9) But to the early history

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(7) See H. v. AVERDUNK, GERHARD MERCATOR (1914), p. 128 sequ. The properties of the MERCATOR projection in the Legenda to the map of 1569. In v. AVERDUNK also discussion of the other literature.

(8) Annotationes JOANNIS VERNERIS, Nuremberg 1514; O. FINAEUS, De linearum, superficierum et corporum dimensionibus (1531). See H. v. AVERDUNK, *l. c.* (7)

(9) See M. CANTOR, Vorlesungen II (1892), p. 840; III, p. 194, see however G. ENESTRÖM, *Bibliotheca mathematica* 12 (1912-13), p. 156; 13 (1913-14), p. 168.

of differential geometry belongs certainly the *Horologium oscillatorium* of CHRISTIAEN HUYGENS (1629-1695), his book on pendulum clocks (1673) (10). The problem of measuring time in an exact way suggested here a new mathematical theory. One of the chapters of the book gives a complete theory of evolutes and involutes in the plane. HUYGENS wanted a pendulum so constructed that the period of vibration would be independent of the altitude. This is the problem of the tautochrone. The solution is that the mass of the pendulum moves, not on a circle, but on a cycloid. But the evolute of the cycloid is another cycloid. We can therefore get a tautochronic pendulum by forcing the thread of the pendulum to move along the circumference of two small parts of a cycloid with cusp at the point of suspension and cusp tangent in the direction of equilibrium. To find this form of the " cheeks " HUYGENS develops the general theory of evolutes and involutes (" evolutae " and " evolventes", as he calls them), in the plane and he gets an expression, in geometrical form, for the radius of curvature. Here we also find the theorems that the involute intersects orthogonally the tangents of the evolute and the relation between arc-length of involute and length of the tangent to the evolute.

## 2. — *The First Systematic Contributions*

When LEIBNIZ started his work, analytical geometry of the plane was well under way, as was the application of infinitesimals to quadratures. His main contributions to differential geometry can be found in papers of 1684, 1686 and 1692.

In the *Nova methodus pro maximis and minimis*, the first paper in which LEIBNIZ published his new method (*Acta Eruditorum* 1684), we already find the interpretation of the equation $d^2y = 0$. It indicates a point of inflexion, a conception, as we saw, introduced by DESLUSE and FERMAT. In a paper of 1686 (11), we find the circle of osculation, but not the expression for its radius in analytical form. LEIBNIZ thought in his paper that the circle of osculation passes through four consecutive points of the curve, because

(10) German translation in *Ostwald's Klassiker*, 192.

(11) LEIBNIZ, Meditatio nova de natura anguli contactus et osculi. *Math. Schriften*, ed. GERHARDT II 3, p. 326-329.

it has two contacts with it. After a remark by JACOB BERNOULLI (1692, *Acta Eruditorum*) he readily recognised that only three consecutive points come into consideration. The word " osculation " is taken from this paper of 1686.

In 1692, LEIBNIZ published (12) his theory of envelopes of a family of plane curves $f\ (x, y, \alpha) = 0$. It is necessary for this to eliminate between $f = 0$ and $\frac{\delta f}{\delta \alpha} = 0$. This early result is the more remarkable, as only in recent times some essential advance is made on this statement. In another paper of 1692 (13) we find a discussion of evolutes and involutes, mainly a statement of HUYGENS' results, with the additional remark that the different involutes are " parallel," the first place where this word is used for plane curves.

To what extent differential calculus was applied to geometry in those early days of the new method can be estimated by the recently published lectures of JOHANN BERNOULLI at Basle in the winter of 1691-92. (14) There we find computation of tangents to plane curves, with cycloid, cissoid, quadratrix, as examples. Maxima and minima are found by taking $dy = 0$. The condition $ddy = 0$ leads to points of inflexion, as shown for the case of conchoid and versiera. Even polar coordinates in the plane are introduced. In the Integral calculus, written at the same time, the radius of curvature appears (14). L'HOSPITAL wrote his *Analyse des infiniments petits* (1696), the first published textbook on the calculus, under the influence of these lectures of BERNOULLI; L'HOSPITAL did not add much of interest to us.

With the entrance of the BERNOULLI brothers into the field a highly competitive race for new results begins. The principal figures become engaged in bitter quarrels, LEIBNIZ against NEWTON, JOHANN against JACOB BERNOULLI. The net result for science was a development of such rapidity that even modern times can

(12) LEIBNIZ, De linea ex lineis numero infinitis ordinatim ductis inter se concurrentenbus formata easque omnes tangente. *Acta Eruditorum* 1692. LEIBNIZ *Math. Schriften*, ed. GERHARDT, II 1, p. 266-269.

(13) LEIBNIZ. Geneialia de natura linearum, anguiloque contactus et osculi. *Math. Schriften* 2e Abh. III, p. 331-337.

(14) Differential calculus : *Ostwald's Klassiker* 211, Integral Calculus (Opera III, p. 386) : *Ostwald's Klassiker* 194.

scarcely break the record. Before 1700 many new curves are discovered, many ordinary differential equations are solved, the first elliptic integral is introduced, and the calculus of variations is set up. In 1697 and 1698, the BERNOULLIS study geodesic lines on a surface; JOHANN discovers that osculating plane and tangent plane are perpendicular : " quod planum transiens per tria quaelibet puncta proxima lineae quaesitae debeat esse rectum ad planum tangens superficiem curvam in aliquo istorum punctorum." (15) The equation of the geodesic lines does not appear either in print or in private letters, though JOHANN claims that he has found it. (16) JACOB also outlines an inquiry into the so-called isoperimetrical problems. Both brothers investigate orthogonal and more general trajectories in the plane. The name " trajectory " occurs in a letter of JOHANN to LEIBNIZ of 1698. (17) An application of this theory was found in the theory of light in a medium of varying density, under HUYGENS' assumption that a ray of light intersects the wave front orthogonally. An application of JACOB lies in the finding of the orthogonal trajectories of logarithmic curves.

The problem was taken up again in 1716, when LEIBNIZ, in the priority quarrel, tried to induce NEWTON to show the power of his methods. He asked NEWTON (via CONTI) to find the orthogonal trajectory to a given set of curves, for instance, all hyperbolas of equal center and vertex. NEWTON answered, but only in a general way and his answer does not suggest the best method of attack. He seems to indicate that the finding of orthogonal trajectories depends on the determination of their center of curvatures as intersections of consecutive normals to the given curves; this suggests a differential equation of the second order instead of the first. (18)

This brings us to the question of the contributions of NEWTON to the application of analysis to geometry. Here we are unable to find much worth mentioning except his general method. If

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(15) For the literature see M. CANTOR, Vorlesungen III (1908), p. 229, 232, 235. The quoted passage in a letter to LEIBNIZ of August 1698.

(16) In letters to LEIBNIZ, see G. ENESTRÖM, Sur la découverte de l'équation générale des lignes géodésiques, *Bibliotheca mathematica* 13 (1899), p. 19-24.

(17) JOH. BERNOULLI, Opera I, p. 266, see CANTOR III, p. 222, 233, 443-445.

(18) See CANTOR III, p. 444, 445.

he really wrote his *Theory of Fluxions*, published in 1736, as early as 1671, he was the first to find an analytical expression for the radius of curvature of a plane curve. But this is doubtful. (19)

This dearth of investigations on differential geometry, which continued in England long after NEWTON's death and even now has not disappeared, is the more remarkable as the method of fluxions was geometrical. NEWTON's reasoning was always geometrical, the algorithms belonging to LEIBNIZ. But even later, in MACLAURIN's *Theory of Fluxions* (1742), in which hardly any formulae are used, we do not get more differential geometry than the old theory of curvature for plane curves. The justifying claim was that the book established more exact foundations of the Newtonian way of reasoning.

We have already reported on certain publications in the early 18th century, but they are isolated and contain no new results. The same may be said of a series of papers by PIERRE VARIGNON (1654-1722) (20) on evolutes and related subjects. Of more importance is a paper by R. A. F. DE RÉAUMUR (1683-1757), which generalized evolutes by considering lines intersecting a plane curve under arbitrary angle. Then he obtains evolutes which he calls " imparfaites." (21) It was a youthful production of the later thermometrist and investigator of the social life of insects. But this is almost all we can find. After 1700 the interest in differential geometry declines sharply. The young instrument of analysis is used for other purposes. Geometrical problems remain almost untouched for several decades. But the fertile days of LEIBNIZ and the BERNOULLIS achieved a considerable result. We have nearly the whole scheme of elementary differential geometry of plane curves.

## 3. — *The Eighteenth Century*

For many years we have practically the work of two men, but they were great geniuses : CLAIRAUT and EULER.

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(19) *L. c.*, p. 171, 172.

(20) VARIGNON, *Mémoires prés. par div. sav.* 1700-1713.

(21) REAUMUR, Méthode générale pour déterminer le point d'intersection de deux lignes droites infiniment proches qui rencontrent une courbe. *Mém. prés. par div. sav.* 1709, p. 149-161.

ALEXIS CLAUDE CLAIRAUT (1713-1765), when still a boy of sixteen, wrote his *Recherches sur les courbes à double courbure* (1731), which brought him, at eighteen, into the Académie des Sciences. The book is mainly analytic geometry of space, a subject new in those days. Space curves enter as intersection of surfaces, not as independent entities. CLAIRAUT uses algebra, differential and integral calculus in a study of these curves. In differential calculus he considers tangents, subtangents and subnormals. Only those normal lines to space curves are considered that are normal to the surface on which the space curve lies, which implies the knowledge of the existence of the tangent plane to a surface. CLAIRAUT also finds the locus of the points of intersection of the tangents to the space curve with the plane of projection, and the same for the normals.

The integral calculus gives the possibility of rectification and cubature. Here we find at the same time the development of a curve on its projecting cylinder, " si l'on imagine que la surface cylindrique... s'étende le long du plan RAP et se développe pour ainsi dire."

CLAIRAUT's examples are algebraical curves, as the intersection of $y^2 = ax$ and $z^2 = by$, or $x^2 + y^2 = a^2$ and $y^2 + z^2 = a^2$; sometimes he considers a transcendent curve, as the cycloid.

In the last part of the book he asks for the curve on a surface, " décrite en faisant tourner dessus un compas dont une pointe est attachée à un point fixe C," but he does not think so much of geodesic circles as of the intersection of a simple algebraic surface (as the sphere) with a cone. Another set of problems is created by having a curve roll on another (congruent) curve in a plane perpendicular to the plane of this curve, which he specifies for parabola on parabola and circle on circle.

Of importance is the name of the book, through which " courbe à double courbure " became the recognized technical term. CLAIRAUT got the name from HENRI PITOT (1695-1771), in his time a famous hydraulical engineer, who used it in a paper of 1724 dealing with the helix. (22) Neither PITOT nor CLAIRAUT, however, expressed by their choice of the name any knowledge

(22) H. PITOT, Quadrature de la moitié d'une courbe des arcs, appellée la compagne de la cycloide. Histoire de l'Académie de Sciences, 1724, publ. 1726, p. 107-113.

of first and second curvature. " J'ai crû devoir appeller ces sortes de courbes, courbes à double courbure, parce qu'en les considérant de la façon qu'on vient de dire elles participent pour ainsi dire toujours de la courbure de deux courbes, et c'est même le nom qu'on leur donne dans un mémoire de l'Académie Royale des Sciences où on les propose comme un objet digne des recherches des géomètres ", are CLAIRAUT's words.

CLAIRAUT soon became interested in geodetic work, and in a paper of 1733 on this subject he came to the theorem on surfaces of revolution bearing his name; this theorem states that along a geodesic line C

$$\rho \sin \alpha = \text{const},$$

where $\rho$ is the radius of the parallel circle and $\alpha$ the angle of C with that circle (23). Later he came to integrability conditions of differential equations in studies on hydrostatics. It deserves mention as a first step in what we now call PFAFF's problem. (24) He found that

$$Mdx + Ndy + Pdz = 0$$

is exact, when and only when,

$$M\left(\frac{\delta N}{\delta z} - \frac{\delta P}{\delta y}\right) + N\left(\frac{\delta P}{\delta x} - \frac{\delta M}{\delta z}\right) + P\left(\frac{\delta M}{\delta y} - \frac{\delta N}{\delta x}\right) = 0.$$

He actually proves that the condition is not only necessary, but sufficient.

LEONARD EULER's (1707-1783) work is so varied that it is hard, in this outline, to do him justice. From early youth he constantly turned to the application of the calculus to geometry, from work done in 1727 on parallel curves in the plane, (25) intersecting under constant angle to his paper of 1782, on the differential geometry of space curves. In a series of papers between 1728 and 1732 he takes up the problem of the geodesics

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(23) A. C. CLAIRAUT, Détermination géométrique de la perpendiculaire à la méridienne tracée par M. CASSINI, Ib. 1733, publ. 1735, p. 406.

(24) The work dates from 1739 and 1740, see CANTOR III, p. 856, 861. See also CLAIRAUT's " Théorie de la figure de la terre, tirée des principes de l'Hydrostatique. Paris, 1743. German translation in *Ostwald's Klassiker*, no. 189.

(25) See for the literature F. MÜLLER, Über bahnbrechende Arbeiten L. EULERS aus der reinen Mathematik. *Abh. zur Geschichte d. Math. Wiss.* 25 (1907), p. 63-116, esp. 108-113, or CANTOR III and IV.

on a surface. JOHANN BERNOULLI had attracted his attention to it, probably through the aid of his nephew DANIEL, who was at St. Petersburg with EULER. A result was the equation of the geodesics, in the form

$$\frac{Qddx + Pddy}{Qdx + Pdy} = \frac{dx\, ddx + dy\, ddy}{dt^2 + dx^2 + dy^2},$$

where $Pdx = Qdy + Rdt$ connects the variables of the surface. (26) He made applications to several types of surfaces, for instance cones. In 1732 he uses the coordinates $(x, s)$ in a discussion of the cycloid as a tautochrone. In the same number of the *Commentarii* this is also done by G. W. KRAFT. In 1736, in a paper on the tractrix, he introduces among other new coordinates arc length $s$ and radius of curvature $\rho$ as coordinates of a plane curves, and so opens the series of papers on intrinsic geometry. He shows how $x$ and $y$ can be found when $\rho$ and $s$ are given. In the *Mechanics* of 1736 he proves that mass points on a surface without a force field move along geodesics. In 1740 he studies evolutes and involutes, a study leading him, in 1764, to the curious result already announced by JOHANN BERNOULLI, that the $n^{\text{th}}$ involute of a curve for increasing $n$ tends to become a cycloid. With all these results it is rather astonishing that the *Introductio in analysin infinitorum*, the standard textbook EULER published in 1748, contains so little differential geometry. It may have been the intention of EULER to write a special book on this subject; (27) if so, it was never accomplished. The *Introductio* contains only some remarks on singular points and asymptotes of plane curves, and some osculation properties. EULER, writing the equation of plane curves in the form

$$0 = \text{A}\,t + \text{B}\,u + \text{C}\,t^2 + \text{D}\,t\,u + \text{E}\,u^2 + \text{F}\,t^3 + . —$$

A, B, C, D, ... constants

is first led to introduce an osculating conic section at the origin, which he approximates by a parabola, but then changes to the osculating circle. There are also some remarks on concavity and convexity in relation to the ambiguity of the sign of the radius of curvature. (28)

(26) JOHANN BERNOULLI had the equation also in 1728, printed in his Opera IV (1744) p. 108. Here the word " planum osculans " for the first time.

(27) See CANTOR III, p. 784.

(28) L. EULER, Introductio in analysin infinitorum, II, ch. XIV.

It is a remarkable fact that only in 1760 EULER opens an entire new field in differential geometry. All his previous work has been more in the way of elaboration of old results of LEIBNIZ and the BERNOULLIS, with the possible exception of the introduction of natural coordinates. Even CLAIRAUT's work has merit for differential geometry only as a statement of the problems. We must, however, make reservation for EULER's fundamental work on the calculus of variations, culminating in his *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes* of 1744, in which he not only states and gives methods of solutions to isoperimetrical problems, but finds interesting geometrical properties of curves. The best known perhaps is the theorem that the catenoid is a minimal surface. (29)

EULER's paper of 1760, *Recherches sur la courbure des surfaces*, (30) written during his Berlin residence, contains the first important contribution to surface theory, and also to three-dimensional differential geometry in general. So far only the existence of the tangent plane at a point of a surface had been established, and that in not a very satisfactory way (for instance, by CLAIRAUT). EULER here takes a definite step forward, and arrives at the so-called EULER theorem on curvature of surfaces. It states, in EULER's terms)

$$r = \frac{2fg}{f+g+(f-g)\cos 2\alpha},$$

where $f$ and $g$ are the extreme values of $r$, the radius of curvature of a normal section, and $\alpha$ is the angle of this normal section with one of the normal sections of extreme curvature. The form under which we know the theorem is due to DUPIN, but the name " section principale " is due to EULER, as well as the theorem that the two sections of extreme curvature are normal to each other. His demonstration starts with an arbitrary plane section through a point of the surface, then proceeds to an expression of the radius of curvature for this section, the expression being gradually simplified.

Shortly afterwards (1762) LAGRANGE (1736-1813), then a young

(29) L. EULER, Methodus inveniendi V, Ex. VII 47, German translation in *Ostwald's Klassiker*, 46, p. 111.

(30) Histoire de l'Académie royale des Sciences (Berlin), 1760, p. 119-141.

professor at Turin, published his famous paper on the calculus of variations, the main results of which he had already shown to EULER in 1755. In an appendix, he found the differential equations of the minimal surfaces in the form that $p$ and $q$ must be found under the condition that

$$p\,dx + q\,dy, \quad \frac{pdy - qdx}{\sqrt{1 + p^2 + q^2}}$$

are exact differentials. (31)

In 1770 EULER continued his study of surfaces, and began to investigate developables. He represents the $x$, $y$, $z$ of a point on a surface as functions of two variables $t$ and $u$ (the first time the so-called Gaussian variables are introduced), and writes down the conditions that

$$dx^2 + dy^2 + dz^2 = dt^2 + du^2;$$

$$l^2+m^2+n^2=1, \quad \lambda^2+\mu^2+\nu^2=1, \quad l\lambda+m\mu+n\nu=0$$

$$l=\frac{\partial x}{\partial t}, m=\frac{\partial y}{\partial t}, n=\frac{\partial z}{\partial t}; \quad \lambda=\frac{\partial x}{\partial u}, \mu=\frac{\partial y}{\partial u}, \nu=\frac{\partial z}{\partial u}$$

or, as he states, that

" Une considération tout à fait singulière m'a conduit à la solution de ce problème," he writes to LAGRANGE. EULER is able to integrate the equations and to show that the tangents to an arbitrary space curve form such a developable surface. (32) From his integral he does not seem, however, to draw the conclusion that such surfaces are the only real solution.

Differential geometry had advanced thus far when an entirely new development started. With the exception of EULER's papers and occasional work of LAGRANGE, very little had been done

(31) J. L. LAGRANGE, Essai d'une nouvelle méthode pour déterminer les maxima et les minima des formules intégrales indéfinies. *Miscellanea Taurinensia* 1760-61, publ., 1762, p. 173-195. *Œuvres* I, p. 335-362. German translation in *Ostwald's Klassiker*, 47, p. 23.

(32) L. EULER, De solidis quorum superficiem in planum explicare licet. Nova Comm. Petrop. 16, 1771, p. 3-34. See EULER's letters to LAGRANGE of Jan. 16 and March 9, 1770, Œuvres de Lagrange XIV, p. 217, 218, 221-223, 224.

There is another paper of EULER's hand on surface theory, written in this time, between 1766 and 1775, but only published in 1862, Opera posthuma I, p. 494-496 : " Problema invenire duas superficies quarum alteram in alteram transformare licet, ita, ut in utraque singula puncta analoga easdem inter se teneant distantias." Here we find that surfaces are applicable if, in modern notation, $E$, $F$, $G$ are equal, and the remark that a closed surface cannot be bent.

for a long time. Feudalism, in decay, could not send auxiliary forces to the aid of the lone genius. EULER, in many respects, represented this last period of the feudal system, which disappeared intellectually with such undeniable elegance. EULER's creations perhaps may find a counterpart in those of MOZART.

LAGRANGE felt it, "Ne vous semble-t-il pas" he wrote to D'ALEMBERT in 1772, " que la haute géométrie va un peu en décadence ? " He expresses the same view at other places and D'ALEMBERT's answers are sceptical. (33)

We can interpret the new life, which was developing at the military academy of Mézières, as the beginning of the influence of the French revolution on geometry. Here GASPARD MONGE (1746-1818) was professor since 1768, and began in that early time to show that fecundity in geometrical invention which made him the real creator of differential geometry, of descriptive geometry, and directly and indirectly, of modern geometry in general. His starting point was a series of questions on fortification, which led him to descriptive geometry, but he also knew how to use analysis. His first publication, in 1771, already showed the master. It deals with space curves, the first paper on this subject since CLAIRAUT treating this subject for its own sake. (34) It contains a broad exposition of the whole differential geometry of space curves. It is shown how such curves admit an infinity of evolutes, that they all lie on a developable surface, the polar developable, and that they are geodesics of this surface. He also introduces what we would call the rectifying developable and shows that the original curve is a geodesic on this surface. Here appears the normal plane, the radius of first curvature, the osculating sphere. Two types of inflexion exist : inflexion caused by (what we call) torsion zero, and inflexion caused by (what we call) curvature zero. In the first case, the " points de simple inflexion," four consecutive points of the curve lie in one plane, in the second, the " points de double inflexion," three consecutive points lie on a straight line. Several terms, since adopted, appear here

(33) Œuvres de LAGRANGE XIII, p. 229, 232, 237.

(34) G. MONGE, Mémoire sur les développées, les rayons de courbures et les différents genres d'inflexion des courbes à double courbure. *Mém. div. savans* 1785, p. 511-550 (written 1771), also last chapter of the " Applications de l'Analyse à la Géométrie," where only a part is reprinted.

for the first time, as " ligne des pôles," " arête de rebroussement," " developpée." Many applications to plane and space curves illustrate the general theorems.

In 1780 MONGE published a second paper, written in 1775 (35), in which he took up EULER's theory of developables. MONGE intends to simplify EULER's results. But in his hands the whole theory takes another shape. The geometrical part is treated in such a way as to make the great author of the *Recherches sur la courbure des surfaces* and of many more contributions to geometry more analyst than geometer. Nevertheless there is a good deal of the analyst in MONGE. But the formulas always follow the dynamics of geometrical development, so that the integration of a partial differential equation becomes the gradual building up of a geometrical system in space. Nobody except LIE ever equalled MONGE in this direction.

MONGE points out the essential difference between general ruled surfaces and developables, sets up the differential relation $rt - s^2 = 0$ and finds as first integral that there is an arbitrary relation between $p$ and $q$, which means that a developable is always tangent surface to a space curve. It is also the envelope of a two parameter family of planes. Application is made to the tangent developable of two surfaces, which was already partly elucidated in EULER's work, but which as a problem of " ombres et pénombres " had a great attraction for the inventor of descriptive geometry. We also find here the differential equation of the third order for the ruled surfaces, with the solution of the problem of finding the ruled surface passing through three space curves.

The volume of the Mémoires des savans étrangers, of 1785, which contains MONGE's first paper, contains another classic of differential geometry, MEUSNIER's *Mémoire sur la courbure des surfaces*, written in 1776. (36) The title already shows the indebtedness of the author to EULER. MONGE had, indeed,

(35) G. MONGE, Sur les propriétés de plusieurs genres de surfaces courbes, particulièrement sur celles des surfaces développables, avec une application à la théorie des ombres et des pénombres. *Mém. div. savans* IX, 1780, p. 593-624 (written 1775).

(36) *Mém. sav. étrangers* 1785, p. 477-510. An exposition of MEUSNIER's paper not only in CANTOR IV, p. 547-550, but also in DARBOUX, Théorie générale des surfaces I (1887), p. 260-271.

recommended the paper of EULER to one of his pupils, JEAN BAPTISTE MEUSNIER DE LA PLACE (1754-1793), and, working under the direction of his teacher, the young officer not only found EULER's results in a new way, but added the results which for ever carry his name. In one of the principal sections of the surface at a point he draws a circle, tangent to the surface, with radius equal to the " rayon de courbure " at that point in that direction. This circle is rotated about an axis in its plane parallel to the tangent plane and at a distance equal to the second principal radius of curvature. In this way MEUSNIER gets a torus which has the first and second derivatives in common with the surface at the point. Then he takes this torus as representative of the surface and gets not only EULER's theorem, but also " MEUSNIER's theorem," which he interprets with the aid of a sphere tangent to the surface and with radius equal to the normal radius of curvature of the section in the arbitrary direction on the surface.

MEUSNIER uses his torus in finding the condition under which a surface be a minimal surface. LAGRANGE had already found the differential equation. MEUSNIER interprets it by showing that it means that the sum of the radii of principal curvature is constant. Then he interprets this equation by simple geometrical methods in two cases, and finds the twisted helicoid and the catenoid, which for many years were the only minimal surfaces known. EULER already had found the catenoid, but MEUSNIER, it seems, found his solution independently.

This paper remained the only contribution of MEUSNIER to mathematics. He published it under the best auspices; D'ALEMBERT, feeling the new spirit, said " MEUSNIER commence comme je finis." But MEUSNIER went to other spheres of activity, where he also did excellent work. He collaborated with LAVOISIER to separate water in its constituents (paper of 1784) and wrote important papers on the new subject of aeronautics. " Après avoir consacré sa trop courte vie aux recherches les plus neuves, les plus difficiles, les plus fécondes, il a trouvé devant l'ennemi, au siège de Mayence, la mort la plus héroique." (37)

(37) G. DARBOUX, Notice historique sur le général MEUSNIER, 1909. In " Éloges académiques et discours, Paris", HERMANN, 1912, p. 218-262. GOETHE describes how he watched the French soldiers, defeated, leaving Mayence. They carried the body of MEUSNIER away with them.

CHARLES TINSEAU (1749-1822) was also a graduate of the Mézières academy, class of 1771. He presented a paper to the Academy in 1774, which contains, among several fundamental contributions to the analytic geometry of space, the equation of the osculating plane to a space curve, the surface of the tangents to a curve (already introduced by CLAIRAUT), and the theorem that the orthogonal projection of a space curve on a plane has a point of inflexion if its plane is perpendicular to the osculating plane. (38)

In this period falls a charming paper by EULER, in which he investigates what we now call curves of constant breadth, in EULER's terms : " orbiformes." He gets them as involutes of " triangular curves," that are closed curves with three cusps. (39)

In the meantime MONGE had continued his productivity, of which we shall say more in the next chapter. We only mention a paper of 1781, *Mémoire sur la théorie des déblais et des remblais*, (40) which takes as starting point the engineering problem of moving a heap particle after particle from one place to another in a minimum of effort. This leads to line congruences, which admit two sets of developable focal surfaces. When these are normal, the congruence is normal to a surface, and cuts it along the lines of curvature. In this original way the lines of curvature were introduced into literature. Then there is another paper of 1784, in which he integrates the equation of the minimal surfaces. (41) There are more papers, equally fundamental; but as all are collected in his book of 1808, we may discuss them together. Since 1780 he had been living in Paris for six months a year, where he taught hydraulics at the Louvre, but after BEZOUT's death, 1783, he settled there permanently.

About this time the aging EULER again wrote a fundamental

(38) C. M. T. TINSEAU, Solution de quelques problèmes relatifs à la théorie des surfaces courbes et des courbes à double courbure. *Mém. div. savans* IX, 1780, p. 593-624.

(39) L. EULER, De curvis triangularibus. *Acta Petr.* 2 (1778), p. 3-30.

(40) *Mém. div. sav.* 1781 (publ. 1784). See P. APPELL. Mémorial des Sciences mathématiques XXVIII, also our footnote (50).

(41) G. MONGE, Une méthode d'intégrer les équations aux différences ordinaires. *Mém. div. sav.* 1784, p. 118. An improvement was suggested by LEGENDRE, l'Intégration de quelques équations aux différences partielles. *Mém. div. sav.* 1787, p. 311-12.

paper (42). It sets forth the first analytical treatment of the differential geometry of space curves. MONGE had treated the subjects from a geometrical point of view, but had not given an analytical frame. EULER provides for this by taking $x$, $y$, $z$ as functions of the arc length $s$, and the direction coefficients of the three axes of the moving trihedron. For this purpose EULER introduces the spherical image, using the unit sphere as GAUSS did forty years later. The equation of the osculating plane is here given in the symmetrical form $x\,(rdq—qdr) + y\,(pdr—rdp) + z\,(qdp—pdq) = t$, where $t$ is determined by the condition that the plane must pass through a given point of the curve. This symmetrical way of treating coordinates also characterizes other papers of EULER, as, for example, one of 1779, in which he writes the equation of the geodesic lines on a surface,

$$d^2x(qdz\text{-}rdy) + d^2y(rdx\text{-}pdy) + d^2z(pdy\text{-}qdz) = 0,$$
$$\text{where } pdx + qdy + rdz = 0$$

is the differential equation of the surface. In this paper the integration is carried out for rotation surfaces, partially repeating thereby results of CLAIRAUT. (43)

An account of the important work on map projection done in the 18th century, and again through the efforts of EULER and LAGRANGE, supported by LAMBERT is still missing in our report. All this work is carried out after 1770, the time of the revival of differential geometry in general. In 1777, EULER introduced complex numbers in his study of conformal projection, which LAGRANGE, in the same year, used for the more general problem of mapping meridians and parallels of a sphere into an arbitrary orthogonal system of plane curves. There is a discussion of this work by V. KOMMERELL in CANTOR's *Vorlesungen* IV, p. 572-576 and in books on cartography. Here, on p. 508-511, is also a discussion of work on parallel curves done in the same period (EULER, NIEUPORT, KÄSTNER, and others).

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(42) L. EULER, Methodus facilis omnia symptomata linearum curvarum non in eodem plano sitarum investigandi. *Acta Petr.* 1782, I, p. 19-57 (publ. 1786).

(43) L. EULER, Accuratior evolutio problematis de linea brevissima in superficie quacumque ducenda, *Nov. Act. Petr.* XV, 1779, p. 44-54.

## 4. — *Monge and the École Polytechnique.*

The French Revolution influenced scientific thought in all directions. Under its influence modern geometry was born. In algebraic geometry the strict prescriptions of Greek thought were discarded, and an entirely new school of reasoning was created. In differential geometry the mathematicians at last learned to apply the century-old thoughts of LEIBNIZ and the BERNOULLIS and to establish a science on collective work where EULER so long had pioneered alone.

GASPARD MONGE was uncontested leader. His political work was reflected in his scientific activity. During the revolution he joined the Jacobins, but like many of his political friends he later supported the Empire, which they interpreted as the executor of the will of the French Revolution. NAPOLEON entrusted to MONGE many important functions; he even made him for a while secretary of the navy. On the expedition to Egypt he had with him MONGE as well as many other famous scholars. But MONGE's life work became the organization and scientific leadership of the Ecole Polytechnique, of which he was the director from its beginning, in 1794, till the fall of the Empire. But the Restoration and the old Republican were irreconcilable, and MONGE had to resign. He died a few years later, closing a life not only crowded with scientific achievements but characterized by a unity of thought and deeds seldom found among scholars.

The importance of the Ecole Polytechnique for the development and the organization of science has so well been treated by FELIX KLEIN in his lectures on the history of mathematics in the 19th century (44), that we need not discuss it here. From the beginning MONGE's teaching was an integrating part of the instruction. Here his remarkable geometrical intuition went hand in hand with practical engineering applications to which his whole manner of thinking was always inclined. He also started a collection of models, later continued by TH. OLIVIER. He taught descriptive geometry as a new subject, and he collected his lessons in the

(44) F. KLEIN, Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert, I. Berlin, SPRINGER, 1926, Ch. II. KLEIN refers to JACOBI's paper. Werke 7, p. 355-370.

*Géométrie descriptive*, which still is a standard treatise. He taught differential geometry in the same way; these lessons appeared first as loose pamphlets, the *Feuilles d'Analyse appliquée à la Géometrie* (since 1795), then, in 1807, with little modification, as a book, *Applications de l'Analyse à la Géometrie*. The lessons show all the characteristics of MONGE's genius. (45)

Those who are interested in a discussion of the *Feuilles d'Analyse* can find the material in KOMMERELL's paper in CANTOR IV. We will here discuss the main line of the *Applications*, which however differ but slightly from the *Feuilles*.

The leading thought of the book is the geometrical interpretation of partial differential equations and the interpretation of geometrical facts into the language of partial differential equations. For this MONGE develops the theory of envelopes, characteristics, and edges of regression. At the same time he shows what the integration process means in space.

As the simplest example let us take Chapter II on cylindrical surfaces. These surfaces can be considered in different ways. If we look at them as surfaces of which the tangent plane is parallel to the generating line and therefore parallel to the direction $x = az$, $y = bz$, we get as the equation $ap + bq = 1$, ($p$ and $q$ are the symbols $p = \frac{\delta z}{\delta x}$, $q = \frac{\delta z}{\delta x}$).

But cylindrical surfaces are also surfaces of which the generating line is always parallel to the line $x = az$, $y = bz$. This gives as equation $y - bz = \varphi\ (x - az)$, where $\varphi$ is an arbitrary function. In this way we get the integral of $ap + bq = 1$. From this we can solve several other problems, as the determination of the cylindrical surface if the direction of the generators and a space curve directing their motion is known, or the determination of a surface that envelops a surface along a given curve. At the same time this shows the manipulations which may be effected with partial differential equation.

In this way every chapter of the *Applications* is built up. MONGE classifies his problems into those leading to partial differential equations of the first order, of the second order, and the third

(45) The most interesting edition is the fifth, with notes of J. LIOUVILLE, 1850. The "Feuilles d'Analyse" appeared again in 1801.

order. To the first category belong the cylinders, the cones, the canal surfaces; to the second, the developables ($rt - s^2 = 0$), the ruled surfaces with generators parallel to a given plane, and to the third category, the ruled surfaces.

But to the second category belongs also that whole class of surfaces whose curvatures satisfy certain simple conditions. MONGE is, starting in 1784, the first to introduce the lines of curvature and their properties (41). In a clever way he integrates (Ch. 16) the lines of curvature on an ellipsoid, by increasing the order of the differential equation. He then solves the following four problems :

1) surfaces with one set of lines of curvature plane (Ch. 17)
2) surfaces with $R_1 = \text{const}$ (Ch. 18)
3) surfaces with $R_1 = R_2$ (Ch. 19)
4) surfaces with $R_1 = - R_2$ (Ch. 20)

Problem 1) leads to the molding surfaces (" surfaces moulures "), 2) to the tube surfaces, 4) to the minimal surfaces. But in 3) MONGE finds the paradoxical result that the sphere alone is a solution. In 3) and 4), he integrates the equations in full, and this shows him the explanation of the paradox. There is, indeed, an extended class of families answering problem 3). But the sphere is the one real surface. All other surfaces are imaginary with one real curve on each. These surfaces, says MONGE, are really curves, with area everywhere zero. This is the first full discussion of imaginaries in geometry.

MONGE, however, does not give any other special cases of his general equations of the minimal surfaces than the two known since MEUSNIER.

Another example of a problem leading to a partial differential equation of the third order is the problem of the spheres of variable radius with centers on a space curve (Ch. 22). If the radius is constant the problem is of the second order, if the curve, besides, is plane, it is of the first. The methods used, in all cases, follow the same line of thought.

The last chapters deal with another problem at which MONGE arrived by noting that the normals to a surface along the lines of curvature form a developable surface. This new problem is the inverse one; to find the surfaces of which the normals are tangent to a given surface.

From these lectures many words remained standard expressions. Besides those mentioned, we have: "lignes de courbure," "enveloppe," "charactéristique," and the notation $p$, $q$, $r$, $s$, $t$ for the partial derivatives. But not only MONGE's differential geometry, also his descriptive geometry has a bearing on our subject. It deals with curves and surfaces, but in a purely constructive way, without formulas. Two ways are thus indicated as methods of attacking geometrical problems on curves and surfaces, the geometrical and the analytical. We see this separation clearly in work of MONGE's pupils, as for example DUPIN, who proves many theorems twice, both by geometry and by analysis. In our present day differential geometry we still show that influence, when we define lines of curvature, asymptotic lines, conjugate lines in two different ways. From this "descriptive" type of geometry, which MONGE taught, projective geometry emanates in the hands of his pupils.

MONGE's general idea of connecting partial differential equations with geometry of space is still a leading method in differential geometry, especially in France. In a modernized way, though only through indirect influence, it dominates the work of SOPHUS LIE.

## 5. — *Monge's Pupils*

A galaxy of brilliant men supported MONGE at the Ecole Polytechnique either as colleagues or as pupils. A school of mathematics was the result, in which analytical geometry and differential geometry flourished, and in which projective geometry was created. A stimulating influence had the necessity of teaching courses on advanced subjects, and there was a regular output of textbooks on mathematics and mechanics, many of which established standards valid till today. Almost all geometry was threedimensional. Our theory of quadric surfaces dates from those times, and a considerable attention was paid to plane and space transformations.

Common to all these scientists was their contact with practice, either in the abstract form of mathematical physics and mechanics, or in the direct form of engineering and economic or political activity. They represented one phase of the emancipation of the French bourgeoisie. NAPOLEON expressed also their ideas when he wrote to LAPLACE : " L'avancement, le perfectionnement des mathématiques sont liés à la prospérité de l'état."

Among the colleagues of MONGE of mature age we find LAGRANGE and LAZARE CARNOT; among his younger colleagues and pupils, FOURIER, AMPÈRE, POISSON, PONCELET, RODRIGUES, LANCRET, CORIOLIS, MALUS, DUPIN, FRESNEL, CAUCHY, SADI CARNOT, SOPHIE GERMAIN. For our purpose we must examine closely the work of AMPÈRE, LANCRET, MALUS, RODRIGUES, and especially that of DUPIN.

A. M. AMPÈRE's (1775-1846) mathematical discoveries are less remembered than his physical, though a certain type of partial differential equation carries his name. He commenced his main physical work only after OERSTED's discovery in 1820 of the influence of the electric current on a magnetic needle, when he was already famous as a mathematician. For us a paper on osculating parabolas (46) is of importance, because it contains the notion (if not the name) of the differential invariant. AMPÈRE recognises the importance of $\rho$ and $s$ as instrinsic coordinates of a plane curve, but remarks that $s$ still depends on an arbitrary constant. Therefore higher derivatives are necessary for truly intrinsic coordinates. He chooses for this the osculating parabola. When its equation with respect to tangent and normal at a point of the curve is $u^2 = pt$, ($p$ the parameter), the curve can be given as a function between $u$ and $t$. He shows that $u, p, t$ are differential invariants under rotations and translations. They depend on third derivatives; and AMPÈRE approaches affine differential geometry, when he finds the condition for points with parabola osculating in five consecutive points (affine curvature zero, as we say now) (47). To similar contact with affine conceptions came LAZARE CARNOT (1753-1823) who, in his *Géométrie de position* of 1803, defined what we now call the affine normal. He also proposed intrinsic coordinates, the radius of curvature, and the angle of affine normal with the ordinary normal, as in AMPÈRE's case a mixture of affine

(46) A. M. AMPÈRE, Sur les avantages qu'on peut retirer, dans la théorie des courbes, de la considération des paraboles osculatrices, avec des réflexions sur les fonctions différentielles dont la valeur ne change pas lors de la transformation des axes (presented 1803). *Journal Ec. Polyt.* 14e cah. (1808), p. 159-181.

(47) See p. 178 of AMPÈRE's paper.

and metric conceptions, which could not be very fertile. (48)

That there were many discussions on intrinsic coordinates in those days is also seen in S. F. LACROIX' much used textbook in three volumes on differential and integral calculus, which in its first volume has more than 250 pages on curves and surfaces. He deals faithfully with all results obtained by EULER, MONGE, MEUSNIER, LAGRANGE and others. He also discusses the suggestions of CARNOT and AMPÈRE, and mentions other ways to study curves independent of their position in the plane. (49)

E. L. MALUS (1775-1812) is famous as the discoverer of the polarization of light (1808). His investigations in optics lead him, as later HAMILTON, to the study of line congruences. This theory dates from MONGE's work of 1781 on " déblais and remblais " (50), but MALUS establishes again with his methods the theorem that in such a congruence each line is in general cut by two other lines, so that the lines are the intersection of two families of developpable surfaces. The application to normals to a surface establishes the theorem on lines of curvature. MALUS studies the behavior of line congruences under reflection and refraction. He also studies what we now call line complexes. Through a mistake he fails to obtain entirely the "MALUS-DUPIN" theorem. (51)

With MICHEL ANGE LANCRET (1774-1807) we have a young, promising scholar, who died too early to fulfil his promises. He belonged with MONGE, FOURIER, AMPÈRE, GEOFFROY ST. HILAIRE and many others, to the scholars who accompanied NAPOLEON on his Egyptian expedition. Later he became a member of the commission appointed to publish the results, but he died at

(48) L. N. M. CARNOT, Géométrie de position. Paris, 1803, 489 p., see Probleme LXXVI, art. 433, p. 477, and art. 432, p. 475-476.

(49) S. F. LACROIX, Traité du calcul différentiel et du calcul intégral. Tome I, Paris. Seconde éd. 1810, 652 p., espec. no. 255, p. 484-485. From AMPÈRE's paper it seems that LACROIX in the first edition of his book started the discussion. The work was continued by GERGONNE, *Annales de mathémathiques*, 4 (1813-14) p. 42-55. GERGONNE, p. 372 of the same volume, in restating some results of DUPIN, frames the term " tangentes principales ".

(50) See note (41), also C. SEGRE, MONGE e le congruenze generali di rette. *Bibliotheca mathematica* 318 (1907-08).

(51) MALUS, Optique, *Jour. Ec. Polyt.* 14 cah., 1808, p. 1-44; Dioptrique, ib., p. 84-129; Traité d'Optique, *Mémoires présentés à l'Institut* 2 (1811), p. 214-302, cont. p. 303-508 as " Théorie de la double réfraction."

33 years of age. He wrote two mathematical papers, (52) the first on the general theory of space curves, the other (published after his death) on " développoides," space curves whose tangents are lines cutting a given space curve under a constant angle different from 90°. His first paper is of a more general nature. It contains the two fundamental quantities of the space curve, which he calls " première flexion " and " seconde flexion." The first is the angle $d\mu$ of two consecutive normal planes, the second the angle $d\nu$ of two consecutive osculating planes. (53) Curvature and torsion appear therefore as differentials, and are not written as finite quantities until CAUCHY. There is a third quantity, the angle $d\omega$ of two " plans rectifians " (this name and conception also appears here), and the " equation of LANCRET " exists:

$$d\mu^2 + d\nu^2 = d\omega^2$$

which shows that only two of the three quantities are independent. As an application we learn that the first flexion of the " développante " is equal to the second flexion of the " développée," and vice versa, a theorem only correct for the differentials.

LANCRET is therefore the first to take up the systematic theory of space curves after EULER, but it seems in an independent way. The line of progress goes here from CLAIRAUT via EULER and LANCRET to CAUCHY and FRENET.

OLINDE RODRIGUES (1794-1851) did some work on lines of curvature, simplifying MONGE's results. A set of important formulas still is called after him (54), and so a formula in the theory of LEGENDRE functions. He found also what we now call the Gaussian curvature of a surface by comparing an element of surface with its spherical image; he missed, however, the " theorema egregium." He later became acquainted with ST. SIMON

(52) M. A. LANCRET, Mémoire sur les courbes à double courbure. *Mémoires présentés à l'Institut* 1 (1806), p. 416-454 (presented 1802); Mémoire sur les développoides des courbes planes, des courbes à double courbure et des surfaces développables, ib., 2 (1811), p. 1-79, presented 1806.

(53) LANCRET says that he got his ideas from FOURIER (p. 420). FOURIER did not publish it himself.

(54) O. RODRIGUES, Recherches sur la théorie analytique des lignes et des rayons de courbure des surfaces, et sur la transformation d'une classe d'intégrales doubles, qui ont un rapport direct avec les formules de cette théorie. *Correspondance sur l'Ec. Polyt.* 3, 1815, p. 162. Also a paper in *Bull. Soc. Philomatique* Paris, (3) 2, 1815, p. 34-36: «Sur quelques propriétés des intégrales doubles et des rayons de courbure des surfaces." Here the name is given as *Rodrigue*.

and did considerable work to propagate the socialistic theories of his master, which prevented him from work in mathematics.

CHARLES DUPIN (1784-1873), since 1824 Baron DUPIN (many eminent mathematicians were given titles of nobility in those days, a custom introduced by NAPOLÉON; MONGE, for instance, became " comte de Pélouse ") is the most important differential geometer among the direct pupils of MONGE. Many of his discoveries were made long before he published them in this books, the delay being partly due to his many duties as a naval officer, some of which carried him " dans des pays presque barbares," such as Corfu. At the age of sixteen he discovered the " cyclide of DUPIN ", (55) about 1807 the " DUPIN theorem " on orthogonal surfaces. His results were collected in the *Développements de géométrie* (1813), later followed by the more applied mathematical *Applications de géométrie et de mécanique* of 1822. (56) The *Développements*, he explicitly stated, have been written as a sequel to MONGE's books. MONGE's two methods of approach, the descriptive and the analytical, appear as different currents in DUPIN's book, half of which is purely geometrical, and half of which uses analysis. These tendencies were soon afterward to grow into entirely different branches, into projective geometry and differential geometry proper. DUPIN therefore in almost all cases proves his theorems in two different ways, geometrically and analytically, again following MONGE who defined lines of curvature geometrically as lines along which the normals form a developable surface and analytically as lines along which the normal curvature has extreme value.

The *Développements* are divided into two sections. The first section contains the theory of the indicatrix, the second of the orthogonal systems of surfaces. The main discoveries of the first section are asymptotic lines and the conjugate sets; that of the second section, " DUPIN's theorem " on the lines of intersection of triply orthogonal systems. This enables the author

(55) See J. BERTRAND, Éloges académiques. Paris, Hachette, 1890, p. 221-246.

(56) CH. DUPIN, Développements de géométrie, avec des applications à la stabilité des vaisseaux, aux déblais et remblais, au défilement, à l'optique, etc. Théorie. Paris 1813.

CH. DUPIN, Applications de géométrie et de mécanique à la marine, aux ponts et chaussées, etc., pour faire suite aux développements de géométrie. Paris, 1822.

to give a new treatment of MONGE's lines of curvature on an ellipsoid by introducing confocal quadrics. The general theory of the indicatrix leads to a discussion of elliptic, hyperbolic and parabolic points, and throws new light upon the lines of curvature and the umbilics. DUPIN even enters into a discussion of the simplest cases of lines of curvature through an umbilic (guided by their behavior on an ellipsoid), and he gives the first geometrical proof of MONGE's theorem that the sphere is the only real surface with only umbilics.

From DUPIN's book date several names, as asymptotic lines, and conjugate directions, and also the modern form of writing EULER's equation,

$$\frac{1}{R} = \frac{\cos^2 \alpha}{R_1} + \frac{\sin^2 \alpha}{R_2}$$

DUPIN continued his research in the *Applications*, where he attacked many problems in applied fields, as stability of floating bodies, optics, and " déblais et remblais." Here, moreover, we also find the correction of "MALUS-DUPIN's theorem" on normal systems of straight lines, and the " cyclide " of DUPIN. (57)

DUPIN lived to a ripe age, but did not continue his work on geometry. His travels led him to many countries, and he became especially interested in the growth of capitalism in England, which he liked to propagate in France. This he did in a great number of papers and books on social subjects. He entered politics, became " pair de France " under the Restauration and " sénateur."

As representative of the purely geometrical school of MONGE in the time of the Restoration we have LOUIS L. VALLÉE (1804-1864), a prominent civil engineer, who wrote a *Traité de géométrie descriptive* in 1819, reprinted in 1825, and dedicated to MONGE. It contains the theory of space curves and surfaces, showing how their theory is built up by geometrical reasoning. Here we find for the first time the word " angle de courbure " together with " angle de torsion." VALLÉE reveals to us why geometry should be studied, quoting MONGE : " Pour faire fleurir l'industrie française, il faut diriger l'éducation nationale vers la connaissance

(57) MALUS-DUPIN's theorem on p. 191, with a criticism of MALUS. See for the history DARBOUX, Surfaces II, p. 280.

des objets qui exigent l'exactitude," and remarks that DUPIN " indique, comme une cause remarquable des succès manufacturiers de l'Angleterre, les soins qu'on donne à l'instruction des ouvriers anglais." (58) It is no accident that we see so many geometers of the Napoleonic time, — MONGE, DUPIN, RODRIGUES, VALLÉE — interested in industrial problems. JACOBI's contention that the only goal of science is the honor of the human mind belongs to a generation already emancipated from the revolution.

We must devote a few remarks to the *Théorie des fonctions analytiques*, in which LAGRANGE, in the first years of the Ecole Polytechnique, tried to build up calculus without the use of infinitesimals (1797). As his main object is the study of " dérivées " $f'(x)$, $f''(x)$, etc. (notation and name appear here for the first time) he is attracted by the contact of curves and surfaces. In this book therefore we find for the first time an elaborate analytical theory of osculation, illustrated by many examples. It served as model to all later expositions of the subject, together with that of CAUCHY.

As the last representative of this school we have therefore to mention A. CAUCHY (1798-1857), who became professor at the Ecole Polytechnique as a royalist, in 1816. He wrote on differential geometry in one of his textbooks (1826) " destiné à faire suite au Résumé des leçons sur le Calcul infinitesimal." (59) This textbook contains a beautiful exposition of the theory as it stood in CAUCHY's time at Paris, and shows in many respects the characteristics of CAUCHY's genius. Like LAGRANGE, another analyst, CAUCHY devotes special attention to the contact of curves and surfaces, and gives a first geometrical definition of the contact of two curves. He takes a point in common with two curves, draws a small circle of radius $i$ with this point as center, considers the angle $w$ intercepted on the circumference by the two curves, and compares $w$ to a power of $i$. As source of inspiration in many details he mentions " les lumières de M. M. AMPÈRE et CORIOLIS." AMPÈRE's investigations on electric currents, indeed, inspired CAUCHY's discrimination between a right-handed and

(58) L. VALLÉE, Traité de géométrie descriptive. Sec. ed. 1825, p. 296-287.

(59) CAUCHY, Leçons sur les applications du calcul infinitésimal à la géométrie. Paris, I (1826), 400 p.; II (1828), 123 p. In this book the word " normale principale " is used (I, p. 285, 298).

a left-handed system of coordinate axes. Other innovations are the use of polar coordinates for the computation of the radius of curvature, the introduction of the " première courbure " $\frac{1}{r}$ and the " seconde courbure " $\frac{1}{R}$ for space curves, a systematic treatment of space curves and the use of the " rayon vecteur." It is entirely clear that CAUCHY used to the fullest extent all previous sources, especially MONGE and DUPIN. (60)

*(to be continued.)*

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(60) Of a certain importance is also the work of TH. OLIVIER (1793-1853), a lieutenant of artillery, who was called to Sweden to found a school after the pattern of the Ecole Polytechnique, and later became professor of descriptive geometry at Paris. He continued MONGE's tradition in his collection of geometrical models, and wrote several papers on differential geometry, e.g., a " Mémoire de géométrie," *Journ. Ec. Polyt.*, cah. 24 (1835), p. 61-91, in which he studies the circular helix having, at a point of a space curve, curvature and torsion equal to the corresponding quantities of the curve.