$$\Gamma^{k}_{ij} = \frac{1}{2}\, g^{lk}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right) \qquad i,j,k,l = 1,2.$$

$$= \frac{1}{2} \sum_{l=1}^{2} g^{lk}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right).$$

$$K = -\frac{1}{g_{11}}\Big(\underbrace{\Gamma^{2}_{12,1} - \Gamma^{2}_{11,2}}_{\text{Símbolos de Christoffel de 2ª clase}} + \Gamma^{k}_{12}\Gamma^{2}_{k1} - \Gamma^{k}_{11}\Gamma^{1}_{k2}\Big)$$

$$\Gamma^{k}_{ij,l} = \partial_l \Gamma^{k}_{ij}. \qquad i,j,k,l = 1,2.$$

Ejemplo: (1) Plano.

$$G = (g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad G^{-1} = (g^{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\Rightarrow \Gamma^1_{11} = \underbrace{\tfrac{1}{2} g^{11}\left(\cancel{\partial_1 g_{11}}^{\,0} + \cancel{\partial_1 g_{11}}^{\,0} - \cancel{\partial_1 g_{11}}^{\,0}\right)}_{l=1} + \underbrace{\tfrac{1}{2} g^{21}\left(\cancel{\partial_1 g_{12}}^{\,0} + \cancel{\partial_1 g_{12}}^{\,0} - \cancel{\partial_2 g_{11}}^{\,0}\right)}_{l=2}$$

$$= 0$$

Similarmente $\Gamma^2_{11} = 0$, $\underset{\displaystyle\Gamma^1_{21}}{\underset{\shortparallel}{\Gamma^1_{12}}} = 0$, $\underset{\displaystyle\Gamma^2_{21}}{\underset{\shortparallel}{\Gamma^2_{12}}} = 0$, $\Gamma^1_{22} = 0$ y $\Gamma^2_{22} = 0$

$$\Rightarrow K = -\frac{1}{g_{11}}\left(\cancel{\Gamma^2_{12,1}} - \cancel{\Gamma^2_{11,2}} + \cancel{\Gamma^k_{12}}\cancel{\Gamma^2_{k1}} - \cancel{\Gamma^k_{11}}\cancel{\Gamma^2_{k2}}\right) = 0.$$

Ejemplo 2 (Cilindro de radio r).

$$x(u,v) = (r\cos u, r\,\text{sen}\,u, v) \qquad u \in (0,2\pi), v \in \mathbb{R}.$$

$$x_u = (-r\,\text{sen}\,u, r\cos u, 0), \qquad x_v = (0,0,1).$$

$g_{11} = E = \langle x_u, x_u \rangle = r^2$

$g_{12} = F = \langle x_u, x_v \rangle = 0$

$g_{22} = G = \langle x_v, x_v \rangle = 1$

$$G = (g_{ij}) = \begin{pmatrix} r^2 & 0 \\ 0 & 1 \end{pmatrix} \qquad G^{-1} = (g^{ij}) = \begin{pmatrix} 1/r^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\Rightarrow \Gamma^k_{ij} = \frac{1}{2} g^{\ell k} \left(\cancel{\partial_i g_{j\ell}}^{\,0} + \cancel{\partial_j g_{i\ell}}^{\,0} - \cancel{\partial_\ell g_{ij}}^{\,0} \right) = 0$$

$$\forall i,j,k = 1,2.$$

Ejemplo 3: (Esfera radio r).

$$\mathbf{x}(u,v) = (r\,\mathrm{sen}\,v\cos u,\ r\,\mathrm{sen}\,v\,\mathrm{sen}\,u,\ r\cos v) \qquad u,v \in (0,2\pi).$$

$$x_u = (-r\,\mathrm{sen}\,v\,\mathrm{sen}\,u,\ r\,\mathrm{sen}\,v\cos u,\ 0)$$

$$x_v = (r\cos v\cos u,\ r\cos v\,\mathrm{sen}\,u,\ -r\,\mathrm{sen}\,v)$$

$$g_{11} = E = \langle x_u, x_u\rangle = r^2\,\mathrm{sen}^2 v \qquad g_{12} = F = \langle x_u, x_v\rangle = 0 \qquad g_{22} = \langle x_v, x_v\rangle = r^2$$

$$G = (g_{ij}) = \begin{pmatrix} r^2\,\mathrm{sen}^2 v & 0 \\ 0 & r^2 \end{pmatrix} \qquad G^{-1} = (g^{ij}) = \begin{pmatrix} 1/r^2\,\mathrm{sen}^2 v & 0 \\ 0 & 1/r^2 \end{pmatrix}.$$

$$\Gamma^1_{11} = \frac{1}{2} g^{11}\left(\cancel{\partial_1 g_{11}} + \cancel{\partial_1 g_{11}} - \cancel{\partial_1 g_{11}}\right) + \frac{1}{2}\cancel{g^{21}}\left(\cancel{\partial_1 g_{12}} + \cancel{\partial_1 g_{12}} - \partial_2 g_{11}\right)$$

$$= 0$$

$$\Gamma^2_{11} = \frac{1}{2} g^{12}\left(\cancel{\partial_1 g_{11}} + \cancel{\partial_1 g_{11}} - \cancel{\partial_1 g_{11}}\right) + \frac{1}{2} g^{22}\left(\cancel{\partial_1 g_{12}} + \cancel{\partial_1 g_{12}} - \partial_2 g_{11}\right)$$

$$= \frac{1}{2} g^{22}\left(-\partial_2 g_{11}\right) = \frac{1}{\cancel{2r^2}}\left(-\cancel{2r^2}\, \text{sen}\, v \cos v\right)$$

$$= -\,\text{sen}\, v \cos v.$$

$$\Gamma^1_{12} = \frac{1}{2} g^{11}\left(\cancel{\partial_1 g_{21}} + \partial_2 g_{11} - \cancel{\partial_1 g_{12}}\right) + \frac{1}{2}\cancel{g^{21}}\left(\cancel{\partial_1 g_{22}} + \cancel{\partial_2 g_{12}} - \cancel{\partial_2 g_{12}}\right)$$

$$= \frac{1}{2} g^{11}\left(\partial_2 g_{11}\right) = \frac{1}{\cancel{2}} \cdot \frac{1}{\cancel{r^2}\, \text{sen}^{\cancel{2}} v}\left(\cancel{2r^2}\, \cancel{\text{sen}}\, v \cos v\right)$$

$$= \frac{\cos v}{\text{sen}\, v} = \cot v.$$

$$\Gamma^2_{12} = \frac{1}{2}\cancel{g^{12}}\left(\cancel{\partial_1 g_{21}} + \partial_2 g_{11} - \cancel{\partial_1 g_{12}}\right) + \frac{1}{2} g^{22}\left(\cancel{\partial_1 g_{22}} + \cancel{\partial_2 g_{12}} - \cancel{\partial_2 g_{12}}\right)$$

$$= 0$$

$$\Gamma^1_{22} = \frac{1}{2} g^{11}\left(\cancel{\partial_2 g_{21}} + \cancel{\partial_2 g_{21}} - \cancel{\partial_1 g_{22}}\right) + \frac{1}{2} g^{21}\left(\cancel{\partial_2 g_{22}} + \cancel{\partial_2 g_{22}} - \cancel{\partial_2 g_{22}}\right)$$

$$= 0$$

$$\Gamma^2_{22} = \frac{1}{2} g^{12}\left(\cancel{\partial_2 g_{21}} + \cancel{\partial_2 g_{21}} - \cancel{\partial_1 g_{22}}\right) + \frac{1}{2} g^{22}\left(\cancel{\partial_2 g_{22}} + \cancel{\partial_2 g_{22}} - \cancel{\partial_2 g_{22}}\right)$$

$$= 0$$

$\Gamma^1_{11} = 0$ $\quad$ $\underline{\Gamma^1_{12} = \Gamma^1_{21} = \frac{\cos v}{\operatorname{sen} v}}$ $\quad$ $\Gamma^1_{22} = 0$

$\underline{\Gamma^2_{11} = -\operatorname{sen} v \cos v}$ $\quad$ $\Gamma^2_{21} = \Gamma^2_{12} = 0$ $\quad$ $\Gamma^2_{22} = 0$

$$K = -\frac{1}{g_{11}}\left(\cancel{\partial_1 \Gamma^2_{12}} - \partial_2 \Gamma^2_{11} + \Gamma^1_{12}\Gamma^2_{11} + \cancel{\Gamma^2_{12}}\Gamma^2_{21} - \cancel{\Gamma^1_{11}}\Gamma^1_{12} - \Gamma^2_{11}\cancel{\Gamma^1_{22}} \right)$$

$$= \cancel{-}\,\frac{1}{r^2 \cancel{\operatorname{sen}^2} v}\left(+ \cancel{\cos^2} v - \cancel{\operatorname{sen}^2} v - \cancel{\cos^2} v \right)$$

$$= \frac{1}{r^2}$$

Γ^2_{11} $\qquad$ $\Gamma^1_{12} = \Gamma^1_{21}$