Computer vision, short review of Euclidean geometry

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1 2D Euclidean geometry

1.0.1 Rigid transformations in the Euclidean plane

An **isometry** (or **rigid transformation**) is a transformation that leaves the distance between points **invariant**. What kind of transformations in the plane satisfy this property?

- Translations.
- Rotations.
- Reflections with respect to straight lines.
- Combinations of the previous ones.

In practice, in computer vision, we will handle **translations** and **rotations** (the reflections, changing the orientations, are not common).

Translation in the Euclidean plane (2 parameters t_x, t_y):

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix}$$

or

$$\mathbf{p}' = \mathbf{p} + \mathbf{t}.$$

Rotation in the Euclidean plane, centered on the origin (1 parameter θ):

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

or

$$\mathbf{p}' = \mathbf{R}_\theta \mathbf{p}.$$

where $\mathbf{R}_\theta$ is the 2×2 rotation matrix of angle θ.

Properties of rotation matrices:

- $\mathbf{R}_\theta^T \mathbf{R}_\theta = \mathbf{I}_{2\times 2}$
- $det(\mathbf{R}) = 1$ (this is the difference with reflections!).

The **columns** are the images by the transformation of the vectors of the canonical base: This allows to verify signs...

Example: What are the transformations of the plane whose matrix representations are:

- $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.
- $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

The composition of the previous two types of transformations give the general form of the **direct** isometries in the plane (those that preserve the orientation, excluding the reflections):

$$\mathbf{p}' = \mathbf{R}_\theta \mathbf{p} + t.$$

The set of isometries in the plane equipped with the operation of composition (i.e., apply one transformation after the other) is a group, the **Euclidean group** E(2).

The set of direct isometries in the plane, equipped with the same operation, is a sub-group of E(2) called the **Special Euclidean group**, SE(2). The transformations are also called **displacements**.

Neutral element? Inverse? Associativity? Commutativity?

Observe that, algebraically, we have heterogeneous operations: one is an addition between vectors, and the other a matrix multiplication (affine form). To handle everything in a linear way only, with matrix operations, is to use **homogeneous coordinates**. For the moment, consider that it would just work by replacing:

$$\begin{pmatrix} x \\ y \end{pmatrix} \text{ by } \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

How do translations and rotations can be written with these coordinates?

Translations:

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Rotations:

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

and displacements, in general:

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

The rigid transforms in the plane can be represented by 3×3 **matrices** acting over homogeneous coordinates. We will see that this can be generalized to **projective transformations** (perspective).

```
In [ ]: import cv2
        import math
        import numpy as np
```

```
img = cv2.imread('imgs/messi5.jpg',0)
rows,cols = img.shape

theta = 0.2
fac   = 1.0
M = np.float32([[fac*math.cos(theta),-fac*math.sin(theta),10],[fac*math.sin(theta),fac*m
dst = cv2.warpAffine(img,M,(cols,rows))

cv2.imshow('img',dst)
cv2.waitKey(0)
cv2.destroyAllWindows()
```

1.0.2 Straight lines

General form of the equation of a straight line:

$$ax + by + c = 0$$

Observe that in **homogeneous coordinates**, one can write:

$$a.x + b.y + c.1 = 0$$

which looks like a relation of **orthogonality**:

$$\mathbf{l}^T \mathbf{p} = 0$$

and we can see $\mathbf{l} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ as a representation of the straight line.

Observation: $\begin{pmatrix} 2a \\ 2b \\ 2c \end{pmatrix}$ can also be used! (or any scaled version).

A bit later, we will come back to the notion of **duality** between points and straight lines in the plane.

2 3D Euclidean geometry

2.0.1 Rigid transformations in the 3D space

Similarly as in the 3D case: the isometrie are the transformations that preserve Euclidean distances. They form a group called the Euclidean grou E(3). Those that also preserve the orientations are:

- The 3D *rotations*.
- The 3D *translations*.

We will also handle homogeneous coordinates, but this time with **4** coordinates:

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{R} & & & \mathbf{t} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

The 3D rigid transformations can be represented as 4×4 **matrices** acting on homogeneous coordinates.

- $\mathbf{t}$ is a 3D translation vector.
- $\mathbf{R}$ is a 3×3 rotation matrix.

Observe that $\mathbf{t}$ is the coordinate vector of the **image of the frame origin**.
Properties of 3D rotation matrices:

- $\mathbf{R}^T\mathbf{R} = \mathbf{I}_{3\times 3}$
- $det(\mathbf{R}) = 1$.
- They can be parameterized by elementary rotations along consecutive axis; for example (but there are **many more ways to do it**):

$$\mathbf{R} = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

One needs to specify well the 3 consecutive rotation axis along whith the elementary rotations are applied.

Euler representation.

This representation has a singularity: see what happens when $\theta = 0$: *gimbal lock* (at some particular configurations, one loses a degree of freedom and the possibility to reach all the neighbouring rotations).

Example: What are the transformations of the plane whose matrix representations are:

- $\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$.

2.0.2 Invert transformations

Observe that with the homogeneous matrix representations, it is quite simple to invert 3D displacements:

$$\begin{pmatrix} \mathbf{R} & & & \mathbf{t} \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{R}^T & & & -\mathbf{R}^T\mathbf{t} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Invert a rigid transform is equivalent to apply the inverse of the rotation($\mathbf{R}^T$) and translation $-\mathbf{R}^T\mathbf{t}$.

2.0.3 Plane equations

General form of the equation of a plane:

$$ax + by + cz + d = 0$$

Observe that in **homogeneous coordinates**, this can be seen as:

$$ax + by + cz + d.1 = 0$$

i.e., again, as an **orthogonality** relation:

$$\mathbf{ß}^T \mathbf{p} = 0$$

and we can see $\mathbf{ß} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ as a representation of the plane.

Observation: $\begin{pmatrix} 2a \\ 2b \\ 2c \\ 2d \end{pmatrix}$ can also be used! (or any scaled version).

2.0.4 Cross product

Let two 3D vectors:

$$\mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

and

$$\mathbf{v}' = \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix}$$

the cross-product of $\mathbf{v}$ and $\mathbf{v}'$ is denoted as $\mathbf{v} \times \mathbf{v}'$ and it is defined as the 3D vector:

$$\mathbf{v} \times \mathbf{v}' = \begin{pmatrix} bc' - b'c \\ a'c - ac' \\ ab' - a'b \end{pmatrix}.$$

The obtained vector is perpendicular to both $\mathbf{v}$ and $\mathbf{v}'$: this allows to get an easy expression of the normal of a plane specified through 3 points $\mathbf{A}, \mathbf{B}, \mathbf{C}$:

$$\mathbf{n} = (\mathbf{B} - \mathbf{A}) \times (\mathbf{C} - \mathbf{A})$$

Observe that we have the equivalence:
two 3D vectors $\mathbf{v}$ and $\mathbf{v}'$ are colinear iff $\Leftrightarrow \mathbf{v} \times \mathbf{v}' = 0$.

2.0.5 Straight lines in the 3D space

How many parameters to represent a straight line?

5 parameters: 1 particular point (3 parameters) and one direction (2 parameters)

The most commonly used representation is the one of Plucker coordinates: let $\mathbf{v}$ be a direction vector for the line, and $\mathbf{p}$ one point belonging to it. The Plucker coordinates are:

$$(\mathbf{v}, \mathbf{p} \times \mathbf{v})$$

- A point $\mathbf{q}$ belongs to the line:

$$\mathbf{q} = \mathbf{p} + \lambda \mathbf{v} \Leftrightarrow \mathbf{q} \times \mathbf{v} = \mathbf{p} \times \mathbf{v}$$

- Check by yourself that ($\mathbf{p} \times \mathbf{v}$) is **independent** of the election of $\mathbf{p}$.
- Also observe the projective nature of the representation: you get the same object when using the representation scaled by a non-zero scalar.

2.0.6 Skew matrix associated to the cross product

Observe that if one sees the operator:

$$\mathbf{v} \times \mathbf{x}$$

as an unary operator on $\mathbf{x}$ (with $\mathbf{v}$ fixed), then it is a **linear** operator in $\mathbf{x}$.

Then we can represent the linear map:

$$f_{\mathbf{v}}(\mathbf{x}) = \mathbf{v} \times \mathbf{x}$$

in a matrix form:

$$f_{\mathbf{v}}(\mathbf{x}) = \mathbf{A}(\mathbf{v})\mathbf{x}$$

One can check that:

$$\mathbf{A}(\mathbf{v}) = \begin{pmatrix} 0 & -c' & b' \\ c' & 0 & -a' \\ -b' & a' & 0 \end{pmatrix}.$$

The matrix $\mathbf{A}(\mathbf{v})$ is denoted as $[\mathbf{v}]_{\times}$.

It is **skew-symmetric**.